Abstract

This note elaborates on the development presented in Chapter 3 of *The Theory of Scintillation with Applications in Remote Sensing.* The links to the source of the development, a paper by L. C. Lee, and the most commonly cited secondary source of that material, a paper by K. C. Yeh and L. C. Liu, are discussed in detail. Definitions not included in the book are introduced to clarify the relations among the various realizations of Lee's result. Confusion has arisen because the basic equations can be written in term of the refractive index, the dielectric constant, and ionospheric electron density, which has been compounded by definitions presented in mks units and cgs units.

1 Field Moment Computation–Review

The source of the moment equations that capture the statistical theory of scintillation (Rino [1, Chapter 3]) is the paper by L. C. Lee [2]. Lee's result was summarized in the widely cited review paper by Yeh and Liu [3]. However, Lee's development is difficult to follow, and an attempt was made in Rino [1, Chapter 3] to construct a procedure that could be used to reproduce the Lee's result directly from the forward propagation equation (FPE). The FPE is written in terms of the refractive index rather than electron density, which was used in the Yeh and Liu review. The FPE also uses the x-axis rather than the z-axis as a reference.[1]

For comparison purposes Lee's Equation (37) is rewritten here with the x-axis as the propagation reference and correlation products restricted to two frequencies with equal numbers of displaced correlation variables:

$$
\begin{aligned}
&\frac{\partial \Gamma_{NN}(\cdots)}{\partial x} = \\
&\frac{i}{2}\left(\frac{1}{k_1}\left[\nabla_1^2 + \cdots \nabla_N^2\right] - \frac{1}{k_2}\left[\nabla_{1'}^2 + \cdots \nabla_{N'}^2\right]\right)\Gamma_{NN}(\cdots) \\
&-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{A\left(\boldsymbol{\zeta}_i - \boldsymbol{\zeta}_j\right)}{k_1 k_2}\Gamma_{NN}(\cdots) \\
&-\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{2\left[A\left(\boldsymbol{\zeta}_i - \boldsymbol{\zeta}'_k\right) + A\left(\boldsymbol{\zeta}'_i - \boldsymbol{\zeta}_k\right)\right]}{k_1 k_2}\Gamma_{NN}(\cdots) \\
&-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{A\left(\boldsymbol{\zeta}'_i - \boldsymbol{\zeta}'_k\right)}{k_1 k_2}\Gamma_{NN}(\cdots).
\end{aligned}
\tag{1}
$$

The ellipsis represents the collection of displaced independent variable pairs where the field samples are taken. The equation is also written with the full diffraction operator in place of its parabolic approximation. For dimensional

[1] The x-axis was chosen to distinguish nearly horizontal propagation over a boundary surface from surface scatter theory, which uses the nadir direction as a reference.

consistency, the term $A(\boldsymbol{\zeta})/k^2$, which comes from Lee's papaer, must have units of reciprocal length. Thus, the units of Lee's correlation function, $A(\boldsymbol{\zeta})$, must be l^{-3}.

To pursue this, it is necessary to consider the original Helmholz equation in the forms used respectively by Yeh and Liu and Lee:

$$\nabla^2\psi + k^2\epsilon\psi = 0 \tag{2}$$

$$\nabla^2\psi + k^2\left[1+\beta/k^2\right]\psi = 0. \tag{3}$$

The equivalence $\beta/k^2 = \delta\epsilon$ is unique to Lee's development. The relation between $\delta\epsilon$ and δN_e comes from plasma physics. The fundamental relation is the definition of the dielectric constant for a cold magnetically biased plasma. In the frequency range of interest for scintillation

$$\epsilon = 1 - \omega_p^2/\omega^2. \tag{4}$$

Lee's definition of the plasma frequency is in cgs units, which introduces an extra 4π and a hidden dependence on ϵ_0, which is unity in the cgs system. For clarity,

$$\omega_p^2 = \begin{cases} \left(4\pi q^2\right)/m_e N_e & \text{cgs} \\ \left(\epsilon_0 e^2\right)/m_e N_e & \text{mks} \end{cases} . \tag{5}$$

Yeh and Liu's paper does not define or use the plasma frequency as an intermediary, but in their book, *Theory of Ionospheric Waves*, Equation (3.3.4) is given in mks units. In mks units

$$\begin{aligned} \epsilon &= 1 - \frac{e^2}{m_e\epsilon_0 c^2}\left(c^2/\omega^2\right)N_e \\ &= 1 - \frac{e^2}{m_e\epsilon_0 c^2}N_e/k^2. \end{aligned} \tag{6}$$

Thus, in mks units Lee's parameter β would be defined as

$$\beta = -\left(e^2/\left(m_e\epsilon_0 c^2\right)\right)N_e. \tag{7}$$

To make the units of β explicit, it is convenient to introduce the classical electron radius

$$r_e = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e c^2}. \tag{8}$$

Thus, in mks units,[2]

$$\epsilon = 1 - 4\pi r_e N_e/k^2.$$

This verifies that $\beta = -4\pi r_e N_e$ has the units l^{-2}. From Lee's paper the definition of $A\left(\boldsymbol{\zeta}-\boldsymbol{\zeta}'\right)$ is

[2] In cgs units the leading factor $1/\left(4\pi\epsilon_0\right)$ is replaced by unity.

$$
\begin{aligned}
A\left(\boldsymbol{\zeta}-\boldsymbol{\zeta}'\right) &= \int_{-\infty}^{\infty} <\beta\left(\Delta x,\boldsymbol{\zeta}\right)\beta\left(\Delta x,\boldsymbol{\zeta}'\right)> dx \\
&= \left(4\pi r_e\right)^2 \int_{-\infty}^{\infty} <N_e\left(\Delta x,\boldsymbol{\zeta}\right)N_e\left(\Delta x,\boldsymbol{\zeta}'\right)> dx. \qquad (9)
\end{aligned}
$$

It follows that $A\left(\boldsymbol{\zeta}-\boldsymbol{\zeta}'\right)$ has the consistent units l^{-3}. Whatever units are used to define r_e, its numerical value and the remaining disposition of 4π factors is fixed.

The relation between $\delta\epsilon$ and the refractive index can be derived as follows:

$$
\begin{aligned}
n &= \sqrt{\epsilon} \qquad (10) \\
n+\delta n &= \sqrt{1+\delta\epsilon} \qquad (11) \\
&\simeq 1+\delta\epsilon/2 \qquad (12)
\end{aligned}
$$

Thus,

$$\delta n = -2\pi r_e N_e/k^2. \qquad (13)$$

Unfortunately, Equation (2.3) in Rino [1] is in error by a factor of two, although working with δn avoided factor-of-two errors in the Chapter 3 development.

To review that development Rino's Equation (3.46) is reproduced here as an intermediate expression for calculating complex field that follows directly from the FPE:

$$
\begin{aligned}
&\frac{\partial\Gamma_{NN}(\cdots)}{\partial x} = \\
&-ik_1\sum_{k=1}^{N}\left\langle\Theta_{\boldsymbol{\zeta}_k}P_{NN}(\cdots)\right\rangle + ik_2\sum_{l=1}^{N}\left\langle\Theta_{\boldsymbol{\xi}_l}P_{NN}(\cdots)\right\rangle \\
&-\left\langle\sum_{n=1}^{N}\left(ik_1\delta n(x,\boldsymbol{\zeta}_n)-ik_2\delta n(x,\boldsymbol{\xi}_n)\right)P_{NN}(\cdots)\right\rangle. \qquad (14)
\end{aligned}
$$

Note that the operations on P_{NN} or it's expectation value, Γ_{NN}, are dimensionally consistent with the reciprocal length units established by the initial derivative. The propagation operator $k\Theta_{\boldsymbol{\zeta}}$ has reciprocal length units as does the product $k\delta n$.

To reproduce Lee's result, the following purely formal relation was introduced:

$$\left\langle\delta n(\mathbf{r})P_{NN}(\cdots)\right\rangle = \left\langle\delta\overline{n}(\boldsymbol{\zeta})\delta\overline{n}(\boldsymbol{\zeta}')\right\rangle\Gamma_{NN}(\cdots). \qquad (15)$$

A rigorous proof of (15) remains to be established, but it's application reproduces a result consistent with the result derived independently by Lee. Reviewing the steps involved, (15) is used to evaluate the media terms in (14). A well established relation between the expectation of derivatives and the associated correlation function for the assumed homogeneous statistics is also used. The following result follows directly from (14) and (15):

$$\frac{\partial \Gamma_{NN}(\cdots)}{\partial x} = -ik_1 \sum_{k=1}^{N} \Theta_{\boldsymbol{\zeta}_k} \Gamma_{NN}(\cdots) + ik_2 \sum_{l=1}^{N} \Theta_{\boldsymbol{\xi}_l} \Gamma_{NN}(\cdots)$$
$$- \left\langle \left(\sum_{n=1}^{N} \left(ik_1 \delta\overline{n}(\boldsymbol{\zeta}_n) - ik_2 \delta\overline{n}(\boldsymbol{\xi}_n) \right) \right)^2 \right\rangle \Gamma_{NN}(\cdots). \tag{16}$$

For comparisons to Lee's result, Rino's result is written in term of autocorrelation functions rather than structure functions:

$$- \left\langle \left(\sum_{n=1}^{N} \left(ik_1 \delta\overline{n}(\boldsymbol{\zeta}_n) - ik_2 \delta\overline{n}(\boldsymbol{\xi}_n) \right) \right)^2 \right\rangle =$$
$$\sum_{n=1}^{N} \sum_{n'=1}^{N} \Big[R_{\delta\overline{n}}(\boldsymbol{\zeta}_{n'} - \boldsymbol{\zeta}_n) \, (k_1)^2 + R_{\delta\overline{n}}(\boldsymbol{\xi}_{n'} - \boldsymbol{\xi}_n) \, (k_2)^2$$
$$- (k_1 k_2) \left(R_{\delta\overline{n}}(\boldsymbol{\zeta}_{n'} - \boldsymbol{\xi}_n) + R_{\delta\overline{n}}(\boldsymbol{\xi}_{n'} - \boldsymbol{\zeta}_n) \right) \Big] \tag{17}$$

In either form, the meaning of the overbar is important. It designates an *infinitesimal* path integration along the propagation reference. The term $\delta\overline{n}(\boldsymbol{\zeta}_n)$ is an abstract construct that has physical meaning only through evaluation of the expectation. The defining relation, which is analogous to Yeh and Liu's definition of $A_{\Delta N}$, is

$$\begin{aligned} \left\langle \delta\overline{n}(\boldsymbol{\zeta}) \delta\overline{n}(\boldsymbol{\zeta}') \right\rangle &= R_{\delta\overline{n}} \left(\Delta\boldsymbol{\zeta} \right) \\ &= \int_{-\infty}^{\infty} R_{\delta n} \left(\eta, \Delta\boldsymbol{\zeta} \right) d\eta, \end{aligned} \tag{18}$$

which has length units, but it appears in the final form of the moment equation scaled by k^2, which preserves the reciprocal length units. Effectively, the overbar distinguishes the incremental path-integral per-unit-length from the correlation of the integrated quantity, which has length squared units.

At this point (16), (17), and (18) completely specify the moment equation in terms of the autocorrelation function of the refractive index. As such, the moment equations can be applied to neutral atmospheric structure or ionospheric structure. The only caveat is that the selection of an integration step size must exceed the characteristic length scale of the structure while remaining infinitesimal. The moment equations themselves contain no explicit length scale.

1.1 Reconciliation with Yeh and Liu 1982

Because of the wide use of Yeh and Liu's Equation (3.39) as the source of the moment equations, it is important to reconcile as well Yeh and Liu's result with Lee's result. Yeh and Liu's Equation (3.39) has the same structure as Lee's

Equation (36), but Yeh and Liu use electron density rather than β. Form the review section

$$\beta = 4\pi^2 r_e^2 R_{\Delta\overline{N}}. \tag{19}$$

Upon making this conversion, the factor $2\pi^2 r_e^2$ in Yeh and Liu's result should be $4\pi^2 r_e^2$. Note also that Yeh and Liu's Equation (2.16), which is optical path not phase as the notation implies, is consistent with the mks definition of r_e. This definition is also consistent with the definition in Table 1 of Rino and Fremouw [4]. We note, however, that the results in Section 2.4 of Bhattacharyya, et. al. [5] are consistent with Lee's result, which indicates that the error in Yeh and Liu's Equation(2.16) was noted.

Regarding the diffraction terms, both Lee and Yeh and Liu use the parabolic approximation. From Equation (2.18) in Rino [1],

$$\begin{aligned} k\Theta &= k\sqrt{1+\nabla_\perp/k^2} \\ &\simeq k+\nabla_\perp/\left(2k\right). \end{aligned} \tag{20}$$

To convert to the parabolic approximation form, $k\Theta$ in (16) is replaced by $\nabla_\perp/\left(2k\right)$, which retains reciprocal length units.

$$\frac{\partial U(x,\boldsymbol{\varsigma})}{\partial x} = \frac{i}{2k}\nabla_\perp U(x,\boldsymbol{\varsigma}) + ik\delta n(x,\boldsymbol{\varsigma})U(x,\boldsymbol{\varsigma}). \tag{21}$$

By comparing the Parabolic Wave Equation (PWE) above to the FPE, it follows that the only formal change in the moment equations is the replacement $k\Theta \leftarrow \nabla_\perp/\left(2k\right)$.

The result in Rino [1] are written in terms of the structure function

$$D_{\delta\overline{N}}\left(\Delta\boldsymbol{\zeta}\right) = 2\left(R_{\delta\overline{N}}\left(0\right) - R_{\delta\overline{N}}\left(\Delta\boldsymbol{\zeta}\right)\right), \tag{22}$$

which accounts for the 1/2 factor in Rino [1] Equation (3.51). The equation is correct and consistent with Lee's result as stated, but an explicit definition of $R_{\delta\overline{N}}\left(\Delta\boldsymbol{\zeta}\right)$, (18), was not provided. Also omitted was the definition of $\delta\overline{\phi}$, although we believe that it is best to avoid a phase equivalent of $\delta\overline{n}$ because the length units are too confusing.

The following development reviews path-integrated phase, which retains radian units. In Section 3.1.5 path integrated phase was introduced in (3.31), which is rewritten here as

$$u(\boldsymbol{\zeta}) = \exp\left\{i\delta\phi\left(\boldsymbol{\zeta}\right)\right\}, \tag{23}$$

where

$$\delta\phi\left(\boldsymbol{\zeta}\right) = k\int_0^{l_p}\delta n\left(\eta,\boldsymbol{\zeta}\right)d\eta \tag{24}$$

Note that

$$\begin{aligned} R_{\delta\phi}\left(\Delta\boldsymbol{\zeta}\right) &= \left\langle\delta\phi\left(\boldsymbol{\zeta}\right)\delta\phi\left(\boldsymbol{\zeta}'\right)\right\rangle \\ &= k^2 l_p\int_{-\infty}^{\infty} R_{\delta n}\left(\eta,\Delta\boldsymbol{\zeta}\right)d\eta, \end{aligned} \tag{25}$$

which has no units.

The power-law spectral densify function (SDF) model used throughout Rino [1] is defined in Section 3.1.4. The Shkarofosky source was introduced in Yeh and Liu, specifically their Equation (2.21). They also introduced the structure function, their Equation (2.23). Although the same SDF model was used, the notation was changed in Rino to better reflect the connection to turbulence theory where scales are defined in the spectral domain. The relevant defining equations are the same, but given the number of parameters that must be carried, it is safer to use refractive index units with appropriate conversions applied as needed.

Following the development above, Equation (3.40) in Rino [1] defines the structure function of the path-integrated phase. The consistent definition of $D_{\delta\overline{n}}(y)$ has no dependence on k^2 or l_p:

$$\begin{aligned} D_{\delta\overline{n}}(y) &= \frac{C_s\Gamma(\nu-1/2)}{2\pi\Gamma(\nu+1/2)} \\ &\quad \times \frac{1-2\left|q_L y/2\right|^{\nu-1/2} K_{\nu-1/2}(q_L y)/\Gamma(\nu-1/2)}{q_L^{2\nu-1}}. \end{aligned} \tag{26}$$

A consistent definition of $R_{\delta\overline{\phi}}\left(\Delta\boldsymbol{\zeta}\right)$ would be

$$R_{\delta\overline{\phi}}\left(\Delta\boldsymbol{\zeta}\right) = k^2 R_{\delta\overline{n}}\left(\Delta\boldsymbol{\zeta}\right), \tag{27}$$

which has the confusing units of reciprocal length. Thus, as noted above, that definition should be avoided.

1.2 Specific Moment Equations

Regarding second-order moments, we believe Equation (3.52) in Rino is correct:

$$\begin{aligned} &\frac{\partial\Gamma_{11}(x,\boldsymbol{\xi},\boldsymbol{\xi}';k_1,k_2)}{\partial x} = -i\left(k_1\Theta_{\boldsymbol{\xi}} - k_2\Theta'_{\boldsymbol{\xi}}\right)\Gamma_{11}(x,\boldsymbol{\xi},\boldsymbol{\xi}';k_1,k_2) \\ &-\left(R_{\delta\overline{n}}(0)\frac{(k_1-k_2)^2}{2} + D_{\delta\overline{n}}(\boldsymbol{\xi}-\boldsymbol{\xi}')k_1k_2\right) \\ &\times\Gamma_{11}(x,\boldsymbol{\xi},\boldsymbol{\xi}';k_1,k_2). \end{aligned} \tag{28}$$

Using the conversion $R_{\delta\overline{n}} = 4\pi^2 r_e^2/\left(k_1k_2\right)^2 R_{\delta\overline{N}}$ this equation becomes

$$\begin{aligned} &\frac{\partial\Gamma_{11}(x,\boldsymbol{\xi},\boldsymbol{\xi}';k_1,k_2)}{\partial x} = -i\left(k_1\Theta_{\boldsymbol{\xi}} - k_2\Theta'_{\boldsymbol{\xi}}\right)\Gamma_{11}(x,\boldsymbol{\xi},\boldsymbol{\xi}';k_1,k_2) \\ &-2\pi^2 r_e^2\left(R_{\delta\overline{N}}(0)(1/k_1-1/k_2)^2 + 2D_{\delta\overline{N}}(\boldsymbol{\xi}-\boldsymbol{\xi}')1/\left(k_1k_2\right)\right) \\ &\times\Gamma_{11}(x,\boldsymbol{\xi},\boldsymbol{\xi}';k_1,k_2), \end{aligned} \tag{29}$$

which differs from Yeh and Liu's Equation (3.42) by the expected factor-of-two when account is taken of the definition of $D_{\delta\overline{N}}$.

Turning to the fourth-order equation, rewriting Rino's Equation (3.64) in terms of $\delta\overline{n}$, rather than $\delta\overline{\phi}$,

$$\frac{\partial\Gamma_{22}(x,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2;\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)}{\partial x} = -\frac{i}{2k}\left[\nabla^2_{\boldsymbol{\zeta}_1}+\nabla^2_{\boldsymbol{\zeta}_2}-\nabla^2_{\boldsymbol{\xi}_1}-\nabla^2_{\boldsymbol{\xi}_1}\right]\Gamma_{2,2}(x,\cdots)$$
$$-\frac{k^2}{2}\left[D_{\delta\overline{n}}(\boldsymbol{\zeta}_1-\boldsymbol{\xi}_2)+D_{\delta\overline{n}}(\boldsymbol{\zeta}_2-\boldsymbol{\xi}_1)+D_{\delta\overline{n}}(\boldsymbol{\zeta}_2-\boldsymbol{\xi}_2)+\right.$$
$$\left.D_{\delta\overline{n}}(\boldsymbol{\zeta}_1-\boldsymbol{\xi}_1)-D_{\delta\overline{n}}(\boldsymbol{\zeta}_1-\boldsymbol{\zeta}_2)-D_{\delta\overline{n}}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\right]\Gamma_{2,2}(x,\cdots). \tag{30}$$

At this point Yeh and Liu introduce a phase structure function defined by their equation (2.24), which differs from Rino's definition by multiplicative factor of two. Moreover, whereas our definition with the overbar explicitly excludes a path-length dependence their definition requires a explicit renormalization by $L = l_p$, which as discussed above can be misleading.

The remaining equations in Sections 3.4 are correct with two exceptions. The complex i should not appear in the exponential arguments of (3.71) and (3.79). Also in (3.79) the k^2 should be replaced by k and the structure functions should be converted to $D_{\delta\overline{n}}$ with the appropriate scaling. A consistent set of definitions in terms of $\delta\overline{n}$ and some other corrections has been prepared as an errata sheet to the book.

References

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