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Notes on the 4th Moment Equation

Definitions:

Index of refraction due to electron density (high frequency limit):

$$\Delta n = \frac{\lambda^2}{2\pi} r_e \Delta N \tag{1.1}$$

Phase in terms of integrated electron density:

$$\phi = k\int_C \Delta n(s)ds = k\frac{\lambda^2}{2\pi} r_e \int_C \Delta N(s)ds = r_e \lambda \Delta N_T \tag{1.2}$$

Phase correlation function:

$$R_{\delta\phi}(\Delta\rho) = k^2 \int_L^L \Delta z \left(1 - \frac{|\tau|}{L}\right) R_{\delta n}(\Delta\rho,\tau)d\tau \rightarrow k^2 L \int_{-\infty}^{\infty} R_{\delta n}(\Delta\rho, z)dz \tag{1.3}$$

Phase structure function:

$$D_{\delta\phi}(\Delta\rho) = 2[R_{\delta\phi}(0) - R_{\delta\phi}(\Delta\rho)] \tag{1.4}$$

Integrated electron density correlation function (*Yeh and Liu*, 1982):

$$A_{\delta N}(\Delta\rho) \equiv \int_{-\infty}^{\infty} R_{\delta N}(\Delta\rho, z)dz = \frac{r_e^2 \lambda^4}{(2\pi)^2} A_{\delta n}(\Delta\rho) \tag{1.5}$$

Integrated refractive index correlation function (*by analogy with Yeh and Liu*, 1982):

$$A_{\delta n}(\Delta\rho) \equiv \int_{-\infty}^{\infty} R_{\delta n}(\Delta\rho, z)dz \tag{1.6}$$

Path integrated refractive index correlation function per unit length (*Rino*, 2011):

$$\left\langle \delta\overline{n}(\rho)\delta\overline{n}(\rho') \right\rangle = R_{\delta\overline{n}}(\Delta\rho) \equiv \int_{-\infty}^{\infty} R_{\delta n}(\Delta\rho, z)dz \tag{1.7}$$

Phase correlation function per unit length $\left\langle \delta\overline{\phi}(\rho)\delta\overline{\phi}(\rho') \right\rangle$:

$$R_{\delta\overline{\phi}}(\Delta\rho) \equiv \int_{-\infty}^{\infty} R_{\delta\phi}(\Delta\rho, z)dz = k^2 R_{\delta\overline{n}}(\Delta\rho) = r_e^2 \lambda^2 R_{\delta\overline{N}}(\Delta\rho) \tag{1.8}$$

Some Useful relationships:

$$\begin{aligned}
R_{\delta\overline{n}}(\Delta\rho) &= A_{\delta n}(\Delta\rho) \\
R_{\delta\overline{N}}(\Delta\rho) &= A_{\delta N}(\Delta\rho) \\
A_{\delta N}(\Delta\rho) &= \frac{r_e^2 \lambda^4}{(2\pi)^2} A_{\delta n}(\Delta\rho) \\
R_{\delta\phi}(\Delta\rho) &= k^2 L A_{\delta n}(\Delta\rho) \\
R_{\delta\phi}(\Delta\rho) &= r_e^2 \lambda^2 L A_{\delta N}(\Delta\rho)
\end{aligned} \tag{1.9}$$

Fourth moment equation expressed in terms of …

the phase correlation function:

$$\frac{\partial}{\partial z}\Gamma_4 = -\frac{i}{k}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\Gamma_4 - \frac{1}{L}\left[2R_{\delta\phi}(0) - 2R_{\delta\phi}(\xi_1) - 2R_{\delta\phi}(\xi_2) + R_{\delta\phi}(\xi_1+\xi_2) + R_{\delta\phi}(\xi_1-\xi_2)\right]\Gamma_4 \quad (1.10)$$

the phase structure function (*Yeh and Liu*, 1982, eqn. 3.51):

$$\frac{\partial}{\partial z}\Gamma_4 = -\frac{i}{k}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\Gamma_4 - \frac{1}{2L}\left[2D_{\delta\phi}(\xi_1) + 2D_{\delta\phi}(\xi_2) - D_{\delta\phi}(\xi_1+\xi_2) - D_{\delta\phi}(\xi_1-\xi_2)\right]\Gamma_4 \quad (1.11)$$

the refractive index correlation function:

$$\frac{\partial}{\partial z}\Gamma_4 = -\frac{i}{k}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\Gamma_4 - \frac{k^2}{L}\left[2R_{\delta n}(0) - 2R_{\delta n}(\xi_1) - 2R_{\delta n}(\xi_2) + R_{\delta n}(\xi_1+\xi_2) + R_{\delta n}(\xi_1-\xi_2)\right]\Gamma_4 \quad (1.12)$$

the phase correlation function per unit length:

$$\frac{\partial}{\partial z}\Gamma_4 = -\frac{i}{k}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\Gamma_4 - \left[2R_{\delta\bar{\phi}}(0) - 2R_{\delta\bar{\phi}}(\xi_1) - 2R_{\delta\bar{\phi}}(\xi_2) + R_{\delta\bar{\phi}}(\xi_1+\xi_2) + R_{\delta\bar{\phi}}(\xi_1-\xi_2)\right]\Gamma_4 \quad (1.13)$$

the phase structure function per unit length (*Rino*, 2011):

$$\frac{\partial}{\partial z}\Gamma_4 = -\frac{i}{k}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\Gamma_4 - \frac{1}{2}\left[2D_{\delta\bar{\phi}}(\xi_1) + 2D_{\delta\bar{\phi}}(\xi_2) - D_{\delta\bar{\phi}}(\xi_1+\xi_2) - D_{\delta\bar{\phi}}(\xi_1-\xi_2)\right]\Gamma_4 \quad (1.14)$$

the refractive index correlation function per unit length:

$$\frac{\partial}{\partial z}\Gamma_4 = -\frac{i}{k}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\Gamma_4 - k^2\left[2R_{\delta\bar{n}}(0) - 2R_{\delta\bar{n}}(\xi_1) - 2R_{\delta\bar{n}}(\xi_2) + R_{\delta\bar{n}}(\xi_1+\xi_2) + R_{\delta\bar{n}}(\xi_1-\xi_2)\right]\Gamma_4 \quad (1.15)$$

the refractive index structure function per unit length (*Rino*, 2011):

$$\frac{\partial}{\partial z}\Gamma_4 = -\frac{i}{k}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\Gamma_4 - \frac{k^2}{2}\left[2D_{\delta\bar{n}}(\xi_1) + 2D_{\delta\bar{n}}(\xi_2) - D_{\delta\bar{n}}(\xi_1+\xi_2) - D_{\delta\bar{n}}(\xi_1-\xi_2)\right]\Gamma_4 \quad (1.16)$$

the integrated refractive index correlation function:

$$\frac{\partial}{\partial z}\Gamma_4 = -\frac{i}{k}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\Gamma_4 - k^2\left[2A_{\delta n}(0) - 2A_{\delta n}(\xi_1) - 2A_{\delta n}(\xi_2) + A_{\delta n}(\xi_1+\xi_2) + A_{\delta n}(\xi_1-\xi_2)\right]\Gamma_4 \quad (1.17)$$

the integrated electron density correlation function (*Bhattacharyya et al.*, 1992):

$$\frac{\partial}{\partial z}\Gamma_4 = -\frac{i}{k}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\Gamma_4 - r_e^2\lambda^2\left[2A_{\delta N}(0) - 2A_{\delta N}(\xi_1) - 2A_{\delta N}(\xi_2) + A_{\delta N}(\xi_1+\xi_2) + A_{\delta N}(\xi_1-\xi_2)\right]\Gamma_4 \quad (1.18)$$

Split Step Solution of the 4th Moment Equation:

Homogeneous slab:

$$\frac{\partial}{\partial z}\Gamma_4 = -\frac{i}{k}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\Gamma_4 + F(\xi_1,\xi_2)\Gamma_4 \tag{1.19}$$

Initial condition (plane wave):

$$\Gamma_4(\xi_1,\xi_2,z_n^-) = I_0^2 \tag{1.20}$$

Structure step:

$$\Gamma_4(\xi_1,\xi_2,z_n^+) = \Gamma_4(\xi_1,\xi_2,z_n^-)\exp\left[F(\xi_1,\xi_2)\right] \tag{1.21}$$

Diffraction step:

$$\hat{\Gamma}_4(\kappa_1,\kappa_2,z_n^+) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma_4(\xi_1,\xi_2,z_n^+)\exp\left[-i(\kappa_1\xi_1+\kappa_2\xi_2)\right]d\xi_1 d\xi_2$$

$$\hat{\Gamma}_4(\kappa_1,\kappa_2,z_{n+1}^-) = \hat{\Gamma}_4(\kappa_1,\kappa_2,z_n^+)\exp\left[-i\frac{\kappa_1\kappa_2}{k}(z_{n+1}-z_n)\right] \tag{1.22}$$

$$\Gamma_4(\xi_1,\xi_2,z_{n+1}^-) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\Gamma}_4(\kappa_1,\kappa_2,z_n^+)\exp\left[i(\kappa_1\xi_1+\kappa_2\xi_2)\right]\frac{d\kappa_1 d\kappa_2}{(2\pi)^2}$$

Scintillation index:

$$S_4^2(z) = \Gamma_4(0,0,z)/I_0 - 1 \tag{1.23}$$

Intensity correlation function:

$$C_I(\xi,z) = \left[\Gamma_4(\xi,0,z)-1\right]/S_4^2(z) \tag{1.24}$$

Integral constraint:

$$\int_{-\infty}^{\infty}\left[\Gamma_4(\xi,0,z_R)-1\right]d\xi = 0 \tag{1.25}$$