Summary of Unbounded Power-Law Phase Screen Results

Charles L Rino
http://chuckrino.com/wordpress/

August, 2013

Abstract

This note summarizes unbounded power-law phase screen results. The computational detalis are presented separately.

1 Introduction

The statistical theory of scintillation presents a hierarchy of first-order differential equations that individually characterize the complex field moments of all orders. A discussion of the specific form of the moment equations and their relation to other published results are discussed in
`http://chuckrino.com/wordpress/wp-content/uploads/2011/05/BookNotesCh32.pdf`
Each equation has two additive terms. One characterizes the free-propagation of the particular moment. The second characterizes the interaction of the particular moment with the refractive index structure. The structure of the refractive index is characterized by structure functions. If the structured region is represented by an equivalent phase screen, the equation for the spectral density function of the field intensity takes a compact analytic form:

$$\Phi_I(\kappa) = \iint \left[\exp\left\{-l_p k^2 g\left(\alpha, \kappa\rho_F^2\right)\right\} - 1\right] \exp\left\{-i\kappa\cdot\alpha\right\} d\alpha. \tag{1}$$

where

$$g\left(\alpha_1, \alpha_2\right) = 8k^2 \iint \Phi_{\delta n}(0,\kappa) \sin^2\left(\kappa\cdot\alpha_{\mathbf{1}}/2\right) \sin^2\left(\kappa\cdot\alpha_{\mathbf{2}}/2\right) d\kappa/(2\pi)^2. \tag{2}$$

Subtracting 1 from the integrand effectively removes the mean intensity. It follows that

$$SI^2 = < I^2 > -1 \tag{3}$$

$$= \iint \Phi_I(\kappa) d\kappa / \left(2\pi\right)^2 \tag{4}$$

2 Structure Model

To accommodate oblique propagation the FPE is rewritten in a continuously displaced coordinate system (CDC). As discussed in Chapter 4 of Rino [1], geometric dependence of the path integral in the CDC system can be accommodated by replacing $\Phi_{\delta n}(0,\kappa)$ with $\Phi_{\delta n}(\tan\theta\widehat{\mathbf{a}}_{k_T}\cdot\kappa,\kappa)$. The effect is to replace κ^2 with a quadratic form. A general inverse-power-law SDF form is constructed as

$$\Phi_{\delta n}(q) = C_s\varphi(q), \tag{5}$$

where

$$\varphi(q) = \begin{cases} q^{-p_1} & q < q_0 \\ q_0^{p_2-p_1}q^{-p_2} & q > q_0 \end{cases}. \tag{6}$$

With some substitutions and manipulations

$$\Phi_I(\mu/\rho_{\mathbf{F}})/\rho_F = \iint \left[\exp\left\{-U^{p_1-2}\gamma\left(\eta,\mu\right)\right\} - 1\right]\exp\left\{-i\mu\cdot\eta\right\}d\eta. \tag{7}$$

where

$$\gamma\left(\eta,\mu\right) = 8\iint\left\{\begin{array}{cc} \varkappa^{-p_1} & \varkappa < q_0 \\ \varkappa_0^{p_2-p_1}\varkappa^{-p_2} & \varkappa > q_0 \end{array}\right. \sin^2\left(\varkappa\cdot\eta/2\right)\sin^2\left(\varkappa\cdot\mu/2\right)d\varkappa/(2\pi)^2 \tag{8}$$

$$U = C_p^{\frac{1}{p_1-2}}\rho_F \tag{9}$$

$$C_p = k^2 l_p C_s \tag{10}$$

2.1 Single Power Law

The two-component power-law model becomes an unconstrained single component power-law when $p_1 = p_2$. If $p_1 = 0$, the transition wavenumber q_0 is effectively an outer scale. The single unconstrained single power-law admits an analytic form for $\gamma\left(\eta,\mu\right)$:

$$\gamma\left(\eta,\mu\right) = C(p)h\left(\eta,\mu\right) \tag{11}$$

where

$$C(p) = \frac{C_p\Gamma(p_1/2-1)}{2\pi\Gamma(p_1/2)\left(p_1-2\right)2^{(p_1-3)}} \tag{12}$$

and

$$h\left(\eta,\mu\right) = 2\left|\eta\right|^{p_1-2} + 2\left|\mu\right|^{p_1-2} - \left|\eta+\mu\right|^{p_1-2} - \left|\eta-\mu\right|^{p_1-2} \text{ for } p_1 \neq 4. \tag{13}$$

The sign change of $\Gamma(p_1/2-1)$ when p_1 exceeds 4 is the only formal change in the result.

One can use the limiting form $K_0\left(x\right) \to -\ln x$ to obtain a result at $p_1 = 4$, but limit is not consistent when evaluated from the result for $p_1 > 4$. Aside from $p_1 = 4$, the result is identical to the result derived by Rumsey [2]. The correct manipulation of the limiting forms of derivatives of modified Bessel functions resolves a long standing disparity noted in [3].

2.2 Two-Dimensional Models

The two-dimensional counterpart to the single power-law model is summarized as follows:

$$\Phi_I(\kappa;U,\rho_F) = \int_{-\infty}^{\infty} \left(\exp\left\{-U^{p-1}\gamma\left(\alpha/\rho_F,\kappa\rho_F\right)\right\}-1\right)\exp\left\{-i\kappa\alpha\right\}d\alpha \qquad (14)$$

$$U = C_p^{\frac{1}{p-1}}\rho_F \qquad (15)$$

$$C_p = k^2 l_p C_s \qquad (16)$$

$$\gamma\left(\eta,\mu\right) = 8\int_{-\infty}^{\infty} C_p\varphi\left(\varkappa\right)\sin^2\left(\varkappa\cdot\mu/2\right)\sin^2\left(\varkappa\cdot\eta/2\right)d\varkappa/(2\pi) \qquad (17)$$

If $\varphi\left(\varkappa\right) = \varkappa^{-p}$,

$$\gamma\left(\eta,\mu\right) = C(p)h\left(\eta,\mu\right) \qquad (18)$$

where

$$C_p(p) = \frac{C_p\ \Gamma\left(\left(3-p\right)/2\right)}{\sqrt{\pi}\Gamma(p/2)\left(p-1\right)2^{p-1}}. \qquad (19)$$

The scalar forms are obtained from the vector forms by replacing the vector variables with their one-dimensional scalar equivalents.

3 Limiting Forms

3.1 Weak Scatter $U \to 0$

With $\gamma\left(\eta,\mu\right)$ specified it is still necessary to use numerical integration to compute $\Phi_I(\kappa;U,\rho_F)$ and SI. However, it can be shown by direct computation that

$$\lim_{U\to 0}\Phi_I(\kappa;U,\rho_F) = 4C_p\varphi(\kappa)\sin^2\left(\kappa^2\rho_F^2/2\right). \qquad (20)$$

For the three-dimensional power-law

$$SI^2 = \frac{C_p\rho_F^{p_1-2}}{\sqrt{\pi}2^{p1/2-1}}\frac{\Gamma\left(\left(6-p1\right)/4\right)}{\Gamma\left(p1/4\right)\left(p1-2\right)} \text{ for } 2 < p1 < 6\ , \qquad (21)$$

which agrees with (3.88) in Chapter 3 [1] when the substitution for $p1$ is made.

3.1.1 Two-Dimensional Model

With the same manipulations for the two-dimensional model

$$SI^2 = \frac{C_p\rho_F^{p-1}}{\sqrt{\pi}2^{(p-1)/2}}\frac{\Gamma\left(\left(5-p\right)\right)}{\Gamma\left(\left(p+1\right)/4\right)\left(p-1\right)} \text{ for } 1 < p < 5.$$

This can be derived from the three-dimensional form by replacing $p1$ by $p1+1$ and multiplying by 2. The doubling comes from the the difference between cylindrical coordinates and assuming no variation along the second axis.

3.2 Strong Scatter $U \to \infty$

With a redefinition of U for the single power-law model,

$$\Phi_I(\mu/\rho_{\mathbf{F}})/\rho_F = \iint [\exp\left\{-\overline{U}^{p_1-2} h\left(\eta,\kappa\right)\right\} - 1] \exp\left\{-i\mu\cdot\eta\right\} d\eta. \tag{22}$$

where

$$\overline{U}^{p_1-2} = U^{p1-2} C(p), \tag{23}$$

and

$$h\left(\eta,\mu\right) = 2\left|\eta\right|^{p_1-2} + 2\left|\mu\right|^{p_1-2} - \left|\eta+\mu\right|^{p_1-2} - \left|\eta-\mu\right|^{p_1-2} \text{ for } p_1 \neq 4. \tag{24}$$

For large U, the contributions to $\Phi_I(\mu/\rho_{\mathbf{F}})/\rho_F$ must come from small values of $h\left(\eta,\mu\right)$, which suggests a Taylor series expansion on the η:

$$h\left(\eta,\mathbf{u}q\right) \simeq 2\left|\eta\right|^{p_1-2} - \frac{\left(p_1-2\right)\left(\eta^2 + \left(p_1-4\right)\left(\mathbf{u}\cdot\eta\right)^2\right)}{q^{4-p_1}}, \tag{25}$$

which agrees with (32) in Rino [3] when substitution for p_1 is made.

For $2 < p1 < 4$, only the first term in (25) servives. It follows that

$$\Phi_I(\mathbf{u}q/\rho_{\mathbf{F}})/\rho_F = \iint [\exp\left\{-2\overline{U}^{p_1-2}\left|\eta\right|^{p_1-2}\right\} - 1] \exp\left\{-i\mathbf{u}q\cdot\eta\right\} d\eta. \tag{26}$$

The integral is not amenable to further analytic simplification, but one can show that

$$SI^2 = \iint \Phi_I(\mu/\rho_{\mathbf{F}})/\rho_F \frac{d\mu}{\left(2\pi\right)^2} = 1. \tag{27}$$

The scintillation can exceed unity with increasing $\overline{U}$, but ultimately it will come back to the Rayleigh limit of unity.

For $4 < p1 < 6$, the second term in (25) dominates. The resulting integral can be evaluated analytically:

$$\Phi_I(\mathbf{u}q/\rho_{\mathbf{F}})/\rho_F = \frac{\pi q^{4-p_1}}{\sqrt{\left(p_1-3\right)}\left(p_1-2\right)2\overline{U}^{p_1-2}} \exp\left\{-\frac{q^{6-p_1}}{4\left(p_1-2\right)\left(p_1-3\right)2\overline{U}^{p_1-2}}\right\} \tag{28}$$

It can also be shown that

$$SI^2 = \frac{2\sqrt{\left(p_1-3\right)}}{6-p_1}. \tag{29}$$

These results agree with (37) in Rino [3], but (39) is a factor-of-two larger. The error was carried to (3.100) in Chapter 3 of Rino [1].

3.2.1 Two-Dimensional Results

Following a parallel development

$$\Phi_I(\kappa;U,\rho_F)=\int_{-\infty}^{\infty}\left(\exp\left\{-U^{p-1}\gamma\left(\alpha/\rho_F,\kappa\rho_F\right)\right\}-1\right)\exp\left\{-i\kappa\alpha\right\}d\alpha. \tag{30}$$

where

$$\gamma\left(\eta,\mu\right)=C(p)h\left(\eta,\mu\right) \tag{31}$$

and

$$h\left(\eta,\mu\right)=2\left|\eta\right|^{p-1}+2\left|\mu\right|^{p-1}-\left|\eta+\mu\right|^{p-1}-\left|\eta-\mu\right|^{p-1}\ \text{ for } p\neq 3. \tag{32}$$

For large U,

$$h\left(\eta,q\right)\ \simeq 2\left|\eta\right|^{p-1}-\frac{(p-1)(p-2)\eta^2}{q^{3-p}} \tag{33}$$

For $1<p<3$

$$\Phi_I(q/\rho_F)/\rho_F=\int[\exp\left\{-2\overline{U}^{p-1}\left|\eta\right|^{p_1-2}\right\}-1]\exp\left\{-iq\eta\right\}d\eta \tag{34}$$

and SI ultimately converges to unity.

For $3<p<5$

$$\begin{aligned}\Phi_I(q/\rho_F)/\rho_F&=\int\exp\left\{-\frac{(p-1)(p-2)2\overline{U}^{p-1}}{q^{3-p}}\eta^2\right\}\exp\left\{-iq\eta\right\}d\eta\\&=\sqrt{\frac{\pi q^{3-p}}{(p-1)(p-2)2\overline{U}^{p-1}}}\exp\left\{-\frac{q^{5-p}}{4(p-1)(p-2)2\overline{U}^{p-1}}\right\}\end{aligned} \tag{35}$$

Similarly,

$$\begin{aligned}SI^2&=2\sqrt{\pi}\int\sqrt{\frac{q^{3-p}}{(p-1)(p-2)2\overline{U}^{p-1}}}\exp\left\{-\frac{q^{5-p}}{4p-1)(p-2)2\overline{U}^{p-1}}\right\}\frac{dq}{2\pi}\\&=\frac{1}{\sqrt{\pi}}\int\exp\left\{-\frac{q^{5-p}}{(p-1)(p-2)2\overline{U}^{p-1}}\right\}d\frac{q^{(5-p)/2}}{\sqrt{4(p-1)(p-2)2\overline{U}^{p-1}}}\\&=\frac{1}{5-p}\frac{1}{\sqrt{\pi}}\int\exp\left\{-u^2\right\}du=\frac{1}{5-p}\end{aligned} \tag{36}$$

This result is also differs by a factor-or-two from (10) in Rino [4].

References

[1] Charles L. Rino. *The Theory of Scintillation with Applications in Remote Sensing*. John Wiley and Sons, Inc., New York, 2011.

[2] V. H. Rumsey. Scintillation due to a concentrated layer with a power-law turbulence spectrum. *Radio Science*, 10(1):107–114, 1975.

[3] C. L. Rino. A power law phase screen model for ionospheric scintillation 2. Strong scatter. *Radio Science*, 14(6):1147–1155, 1979.

[4] C. L. Rino. Numerical computations for a one-dimensional power law phase screen. *Radio Science*, 15(1):41–47, 1980.