Computation of Intensity Spectra for Unbounded power law phase scree.

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Abstract
This note is a recalculation of unbounded power-law phase screen intensity spectra. The analytic results resolve a long standing disparity and provide test cases for numerical analysis procedures that are being used to extend the analysis to two-component power-law spectra.

1 Introduction
The interaction of electromagnetic waves with weakly inhomogeneous structure is characterized by the forward propagation equation (FPE). The formal solution to the FPE is a complex field $\psi(k; x, \varsigma)$ where $k = 2\pi n_0 f/c$ is the wavenumber in a reference background medium with refractive index $n_0$, frequency $f$, and $c$ is the vacuum velocity of light. The coordinate $x$ is the propagation reference direction with $\varsigma$ the normal plane coordinate. The field intensity is

$$I(k; x, \varsigma) = |\psi(k; x, \varsigma)|^2.$$  

(1)

The statistics of $\psi(k; x, \varsigma)$ are characterized by a hierarchy of differential equations, one for each moment of the form$^1$

$$\Gamma_{nm}(x; \zeta_1, \cdots \zeta_N; \xi_1, \cdots \xi_M) = \left( \prod_{n=1}^N \psi(x, \zeta_n) \prod_{m=1}^M \psi^*(x, \xi_m) \right).$$  

(2)

A discussion of the specific form of the moment equations and their relation to other published results are discussed in


2 Phase Screen Model
The phase-screen model assigns the perturbation to an equivalent phase screen formed by a path integral through the structured region. Free-space propagation proceeds from the center of the phase screen to the observer. The average

$^1$See (3.50) and (3.51) in Chapter 3 of Rino [1].
intensity structure is characterized by the fourth-order moment \( \Gamma_{22}(x; \alpha_1, \alpha_2) \). In free space

\[
\frac{\partial \Gamma_{22}(x; \alpha_1, \alpha_2)}{\partial x} = -i \frac{1}{k} \nabla_{\alpha_1} \cdot \nabla_{\alpha_2} \Gamma_{22}(x; \alpha_1, \alpha_2) - k^2 g(\alpha_1, \alpha_2) \Gamma_{22}(x; \alpha_1, \alpha_2),
\]

where

\[
g(\alpha_1, \alpha_2) = D_{\delta n}(\alpha_1) + D_{\delta n}(\alpha_2) - D_{\delta n}(\alpha_1 + \alpha_2)/2 - D_{\delta n}(\alpha_1 - \alpha_2)/2
\]

\[
= 8k^2 \int \Phi_{\delta n}(0, \kappa) \sin^2 (\kappa \cdot \alpha_1/2) \sin^2 (\kappa \cdot \alpha_2/2) d\kappa/(2\pi)^2.
\]

The integral for in (5) follows by direct computation from the definition

\[
D_{\delta n}(\alpha) = 2 (R_{\delta n}(0) - R_{\delta n}(\alpha)) = 2 \int \int (1 - \exp\{i\kappa \cdot \alpha\}) \Phi_{\delta n}(0, \kappa) d\kappa/(2\pi)^2.
\]

The overbar notation \( \overline{\delta n} \) is a reminder of the implicit path integration, which reduces the dimensionality of the structure characterization from three to two.

### 2.1 Fourth Order Moment Solution

Consider the generalized Fourier transformation relation

\[
\hat{\Gamma}_{22}(x, \kappa_1, \kappa_2) = \int \int \Gamma_{22}(x, \alpha_1, \alpha_2) \exp \{-i (\kappa_1 \cdot \alpha_1 + \kappa_2 \cdot \alpha_2)\} d\alpha_1 d\alpha_2.
\]

In the absence of structure \( g(\alpha_1, \alpha_2) = 0 \), whereby (3) reduces to

\[
\frac{\partial \hat{\Gamma}_{22}(x, \kappa_1, \kappa_2)}{\partial x} = i \frac{1}{k} \kappa_1 \cdot \kappa_2 \hat{\Gamma}_{22}(x, \kappa_1, \kappa_2).
\]

In the absence of diffraction \( \nabla_{\alpha_1} \cdot \nabla_{\alpha_2} \Gamma_{22}(x, \alpha_1, \alpha_2) = 0 \), (3) reduces to

\[
\frac{\partial \Gamma_{22}(x, \alpha_1, \alpha_2)}{\partial x} = -k^2 g(\alpha_1, \alpha_2) \Gamma_{22}(x, \alpha_1, \alpha_2).
\]

A formal solution to the fourth-order moment equation can be constructed as follows: Assume that \( \Gamma_{22}(x, \alpha_1, \alpha_2) \) is known. First apply the structure contribution,

\[
\Gamma_{22}^{1/2}(x, \alpha_1, \alpha_2) = \Gamma_{22}(x, \alpha_1, \alpha_2) \exp \{-k^2 g(\alpha_1, \alpha_2) \Delta x\}.
\]

To propagate \( \Gamma_{22}^{1/2}(x, \alpha_1, \alpha_2) \) the generalized Fourier transform is computed. The propagation operator implied by the solution to (8) is then applied:
\[ \hat{\Gamma}_{22}(x + \Delta x, \kappa_1, \kappa_2) = \int \int \left[ \int \int \Gamma^{1/2}_{22}(x, \alpha'_1, \alpha'_2) \right. \\
\times \exp\left\{ -i (\kappa_1 \cdot \alpha'_1 + \kappa_2 \cdot \alpha'_2) \right\}d\alpha'_1d\alpha'_2 \\
\times \exp\left\{ i \kappa_1 \cdot \kappa_2 \Delta x/k \right\}. \] (11)

The new field at \( x + \Delta x \) is reconstructed by computing the inverse Fourier transformation

\[ \Gamma_{22}(x + \Delta x, \alpha_1, \alpha_2) = \int \int \hat{\Gamma}_{22}(x + \Delta x, \kappa_1, \kappa_2) \\
\times \exp\{ i (\kappa_1 \cdot \alpha_1 + \kappa_2 \cdot \alpha_2) \} \frac{dk_1}{(2\pi)^2} \frac{dk_2}{(2\pi)^2}. \] (12)

### 2.2 Intensity SDF

In the transformed coordinate system the intensity correlation function is obtained from \( \Gamma_{22}(x; \alpha_1, \alpha_2) \) by setting \( \alpha_2 = 0 \):

\[ \langle I (k; x, \varsigma) I (k; x, \varsigma') \rangle = \Gamma_{22}(x; \alpha_1, 0). \] (13)

The intensity SDF is the Fourier transformation of \( \Gamma_{22}(x; \alpha_1, 0) \), but forward propagation of the intensity spectrum involves both the \( \alpha_1 \) and \( \alpha_2 \) dependencies. One can show that

\[ \Phi_I(\kappa) = \int \int \Gamma_{22}(x_0, \alpha, \kappa \rho_F^2) \exp\{ -i \kappa \cdot \alpha \} d\alpha, \]

where

\[ \rho_F = \sqrt{x/k}. \] (14)

The split-step solution is simplified significantly if the structured region is replaced by a single equivalent phase screen. At the exit plane of the phase screen

\[ \Gamma_{22}(l_p, \alpha_1, \alpha_2) = \exp\left\{ -l_p k^2 \rho_F^2 (\alpha_1, \alpha_1) \right\}. \] (15)

The subsequent propagation of the intensity SDF in computed as

\[ \Phi_I(\kappa) = \int \int \left[ \exp\{ -l_p k^2 \rho_F^2 (\alpha, \kappa \rho_F^2) \} - 1 \right] \exp\{ -i \kappa \cdot \alpha \} d\alpha. \] (16)

Subtracting 1 from the integrand effectively removes the mean intensity, which is conserved by the FPE. It follows that

\[ SI^2 = \langle I^2 \rangle - 1 \]

\[ = \int \int \Phi_I(\kappa) d\kappa/(2\pi)^2 \] (18)
3 Structure Model

To accommodate oblique propagation the FPE is rewritten in a continuously displaced coordinate system (CDC). As discussed in Chapter 4 of Rino [1], geometric dependence of the path integral in the CDC system can be accommodated by replacing $\Phi_{\delta n}(0, \kappa)$ with $\Phi_{\delta n}(\tan \theta \mathbf{x}_F \cdot \kappa, \kappa)$. The effect is to replace $\kappa^2$ with a quadratic form. A general inverse-power-law SDF form is constructed as

$$\Phi_{\delta n}(q) = C_s \varphi(q),$$  \hspace{1cm} (19)

where

$$\varphi(q) = \begin{cases} q^{-p_1} & q < q_0 \\ q_0^{p_2-p_1} q^{-p_2} & q > q_0 \end{cases}. \hspace{1cm} (20)$$

Note that with $p_1 = 0$, the transition wavenumber, $q_0$, becomes an outer scale. With $p_2 = p_1$, the spectrum is formally an unmodified power law. With $p_2 > p_1$, the $q_0$ weighting ensures continuity at the break point.

With some substitutions and manipulations

$$l_p k^2 g(\alpha, \kappa \rho_F^2) = 8 C_p \rho_F^{-2} \int \varphi(\kappa') \sin^2 (\kappa' \rho_F \cdot (\alpha/\rho_F)/2) \times \sin^2 (\kappa' \rho_F \cdot \kappa F/2) d(\kappa' \rho_F)/(2\pi)^2$$

$$= 8 C_p \rho_F^{p_1-2} \int \varphi(\alpha/\rho_F) \sin^2 (\alpha \cdot (\alpha/\rho_F)/2) \times \sin^2 (\alpha \cdot \kappa F/2) d\alpha/(2\pi)^2 \hspace{1cm} (21)$$

Let

$$\gamma(\eta, \mu) = 8 \int \int \left\{ \begin{array}{l} \eta^{p_1} \eta < q_0 \\ \eta^{p_2-1} \eta > q_0 \end{array} \right\} \sin^2 (\alpha \cdot \eta/2) \sin^2 (\alpha \cdot \mu/2) d\alpha (2\pi)^2.$$ \hspace{1cm} (22)

It follows that

$$l_p k^2 g(\alpha, \kappa \rho_F^2) = U^{p_1-2} \gamma (\alpha/\rho_F, \kappa \rho_F), \hspace{1cm} (23)$$

where

$$U = C_p^{\frac{1}{p_1-2}} \rho_F \hspace{1cm} (24)$$

and

$$C_p = k^2 l_p C_s. \hspace{1cm} (25)$$

$$\Phi_I(\kappa) = \rho F \int \exp \left\{ -U^{p_1-2} \gamma (\alpha/\rho_F, \kappa \rho_F) \right\} - 1 \exp \left\{ -i \kappa \rho_F \cdot \alpha/\rho_F \right\} d\alpha/\rho_F \hspace{1cm} (26)$$

or

$$\Phi_I(\mu/\rho_F)/\rho_F = \int \int \exp \left\{ -U^{p_1-2} \gamma (\eta, \mu) \right\} - 1 \exp \left\{ -i \mu \cdot \eta \right\} d\eta. \hspace{1cm} (27)$$
3.1 Single Power Law

To introduce the single power-law model consider the constrained isotropic form:

\[ R(y) = \frac{C_p}{2\pi \Gamma(p_1/2)} \left( q_0 y / 2 \right)^{p_1/2-1} K_{p_1/2-1} (q_0 y) / q_0^{p_1-2} \]  \hspace{1cm} (28)

\[ \varphi(q) = C_p \left( q_0^2 + q^2 \right)^{-p_1/2} \]  \hspace{1cm} (29)

The unconstrained power-law form is obtained by taking the limit as the outer scale wavenumber \( q_0 \to 0 \). The small argument formula is used to evaluate the modified Bessel functions:

\[ \lim_{x \to 0} K_{\gamma}(x) = \frac{1}{2} \Gamma(|\gamma|) (x/2)^{-|\gamma|} \quad \text{for} \quad \gamma \neq 0. \]  \hspace{1cm} (30)

The absolute values are introduced to ensure that the relation \( K_{\gamma}(x) = K_{-\gamma}(x) \) is preserved. It follows that

\[ \lim_{x \to 0} (x/2)^\gamma K_{\gamma}(x) = \frac{1}{2} \Gamma(|\gamma|) \begin{cases} 1 & \text{for} \quad \gamma > 0 \\ (x/2)^{-2|\gamma|} & \text{for} \quad \gamma < 0 \end{cases}. \]  \hspace{1cm} (31)

The sign of the power-law index determines the limiting form of the product as \( x \to 0 \).

Using the limiting formula with \( p > 2 \), it follows from (28) that

\[ R(0) = \frac{C_p \Gamma(p_1/2 - 1)}{4\pi \Gamma(p_1/2) q_0^{p_1-2}}, \]  \hspace{1cm} (32)

and

\[ D(y) = 2(R(0) - R(y)) = \frac{C_p \Gamma(p_1/2 - 1)}{2\pi \Gamma(p_1/2)} \left( 1 - 2 \left( q_0 y / 2 \right)^{p_1/2-1} K_{p_1/2-1} (q_0 y) / \Gamma(p_1/2 - 1) \right) / q_0^{p_1-2}. \]  \hspace{1cm} (33)

Following L’Hospital’s rule the indeterminate form \( \lim_{q_0 \to 0} D(y) \sim 0/0 \) can be evaluated by taking the limit of the ratio of the derivatives of the numerator and the denominator. Derivatives of the K-type Bessel function terms follow from the relations

\[ \frac{d}{dx} \left\{ x^{\pm|\gamma|} K_{\gamma}(x) \right\} = -x \left\{ x^{\mp|\gamma|-1} K_{|\gamma|+1}(x) \right\}. \]  \hspace{1cm} (34)

For \( 2 < p_1 < 4 \),

\[ D^{(1)}(y) = \frac{-2C_p}{2\pi \Gamma(p_1/2)} \left( \frac{d}{dq_0} \left( q_0 y / 2 \right)^{p_1/2-1} K_{p_1/2-1} (q_0 y) / q_0^{p_1-2} \right). \]  \hspace{1cm} (35)
From (34) with the upper sign
\[
\frac{d}{dq_0} \left( (q_0y/2)^{p_1/2-1} K_{p_1/2-1} (q_0y) \right) = -y (q_0y/2) \left\{ (q_0y/2)^{p_1/2-2} K_{p_1/2-2} (q_0y) \right\}.
\]
Thus,
\[
D^{(1)}(y) = \frac{2C_p}{2\pi\Gamma(p_1/2)} \left( \frac{y (q_0y/2) (q_0y/2)^{p_1/2-2} K_{p_1/2-2} (q_0y)}{(p_1 - 2) q_0^{p_1-3}} \right),
\]
whereby
\[
\lim_{q_0 \to 0} D(y) = \lim_{q_0 \to 0} D^{(1)}(y)
= \frac{C_p \Gamma(2 - p_1/2)}{2\pi \Gamma(p_1/2) (p_1 - 2) 2^{p_1-3}} y^{p_1-2}.
\]
From (6), it follows that the structure function is not defined for an unconstrained power-law with \( p_1 \geq 4 \). However, it follows from (5) that \( g(\alpha_1, \alpha_2) \) is defined for \( 2 < p_1 < 6 \).

To pursue this, note first that from (34) with \( p_1 > 4 \)
\[
\lim_{q_0 \to 0} D^{(1)}(y) = \frac{2C_p}{2\pi\Gamma(p_1/2)} \lim_{q_0 \to 0} \left( \frac{y (q_0y/2) \Gamma(p_1/2 - 2)/2}{(p_1 - 2) q_0^{p_1-3}} \right),
\]
which is indeterminate. However, L'Hospital’s rule can be applied to \( D^{(1)}(y) \) modified to accommodate the sign change of \( p_1/2 - 2 \) when \( p_1 \) exceeds 4:
\[
D^{(2)}(y) = \frac{2C_p}{2\pi\Gamma(p_1/2)} \left( \frac{d}{dq_0} \left( (q_0y/2)^{(p_1/2-2)} K_{p_1/2-2} (q_0y) \right) \right) - y (q_0y/2) \left( (q_0y/2)^{(p_1/2-2)} K_{p_1/2-2} (q_0y) \right)
\]
\[
= \frac{2C_p}{2\pi\Gamma(p_1/2)} \left( \frac{y (q_0y/2)^{(p_1/2-2)} K_{p_1/2-2} (q_0y)}{(p_1 - 2) q_0^{p_1-4}} \right)
\]
\[
\times \left( \frac{d}{dq_0} \left( (q_0y/2)^{(p_1/2-2)} K_{p_1/2-2} (q_0y) \right) \right)
\]
\[
= \frac{2C_p}{2\pi\Gamma(p_1/2)} \left( \frac{y (q_0y/2)^{(p_1/2-2)} K_{p_1/2-2} (q_0y) - y (q_0y/2)^{(p_1/2-2)} K_{p_1/2-3} (q_0y)}{(p_1 - 2) q_0^{p_1-4}} \right).
\]
(40)
Taking the limit as \( q_0 \to 0 \),

\[
\lim_{q_0 \to 0} D^{(2)}(y) = \frac{2C_p}{2\pi \Gamma(p_1/2)} \times \lim_{q_0 \to 0} \left( \left( y^2/2 \right) (q_0 y)/2 \right)^{(p_1/2-2)} \frac{K_{p_1/2-2} (q_0 y) - (q_0 y^2/2) y (q_0 y/2)^{(p_1/2-2)} K_{p_1/2-3} (q_0 y)}{(p_1 - 2) (p_1 - 3) q_0^{p_1-4}}
\]

\[
= \frac{C_p}{2\pi \Gamma(p_1/2) (p_1 - 2)} \times \lim_{q_0 \to 0} \left( \left( y^2/2 \right) (q_0 y)/2 \right)^{(p_1-4)} \frac{\Gamma(p_1/2 - 1) - (q_0 y^2/2) y (q_0 y/2)^{(p_1-5)} \Gamma(3 - p_1/2)}{(p_1 - 2) (p_1 - 3) q_0^{p_1-4}}
\]

\[
= \frac{C_p y^{(p_1-2)}}{2\pi \Gamma(p_1/2) (p_1 - 2) 2^{(p_1-3)}} \frac{\Gamma(p_1/2 - 1) - \Gamma(3 - p_1/2)/2}{(p_1 - 3)}.
\]

(41)

Using the relation

\[
\Gamma(3 - p_1/2) = \Gamma(2 - p_1/2 + 1) = (2 - p_1/2) \Gamma(2 - p_1/2),
\]

(42)

it follows that

\[
\Gamma(p_1/2 - 1) - \Gamma(3 - p_1/2)/2 = \Gamma(p_1/2 - 1) - (2 - p_1/2) \Gamma(2 - p_1/2)/2
\]

\[
= \Gamma(p_1/2 - 1) (1 - (4 - p_1))
\]

\[
= -(p_1 - 3) \Gamma(p_1/2 - 1).
\]

(43)

Making the appropriate substitutions

\[
\lim_{q_0 \to 0} D(y) = \frac{-C_p y^{(p_1-2)} \Gamma(p_1/2 - 1)}{2\pi \Gamma(p_1/2) (p_1 - 2) 2^{(p_1-3)}},
\]

(44)

Whereas, (44) has no meaning as a stand-alone computation, the application of L’Hospital’s rule to \( \gamma (\eta, \mu) \) term by term yields the following result:

\[
\gamma (\eta, \mu) = C(p) h (\eta, \mu)
\]

(45)

where

\[
C(p) = \frac{C_p \Gamma(p_1/2 - 1)}{2\pi \Gamma(p_1/2) (p_1 - 2) 2^{(p_1-3)}}
\]

(46)

and

\[
h (\eta, \mu) = 2 |\eta|^{p_1-2} + 2 |\mu|^{p_1-2} - |\eta + \mu|^{p_1-2} - |\eta - \mu|^{p_1-2} \text{ for } p_1 \neq 4.
\]

(47)

The sign change of \( \Gamma(p_1/2 - 1) \) when \( p_1 \) exceeds 4 is the only formal change in the result.

One can use the limiting form \( K_0 (x) \to -\ln x \) to obtain a result at \( p_1 = 4 \), but limit is not consistent when evaluated from the result for \( p_1 > 4 \). Aside
from $p_1 = 4$, the result is identical to the result derived by Rumsey [2], who
used integration by parts of the isotropic form

$$D_{6\pi}(y) = \int_0^\infty (1 - J_0(\kappa y)) \kappa^{-(p-1)} d\kappa/(2\pi). \quad (48)$$

The correct manipulation of the limiting forms of derivatives of modified Bessel
functions resolves a long standing disparity noted in [3].

### 3.2 Two-Dimensional Models

The two-dimensional counterpart to the single power-law model is summarized
as follows:

$$\Phi_I(\kappa; U, \rho_F) = \int_{-\infty}^{\infty} \exp \{-U^{p-1} \gamma(\alpha/\rho_F, \kappa \rho_F)\} - 1 \exp \{-i\kappa \alpha\} d\alpha. \quad (49)$$

$$U = C_p \frac{1}{\rho_F} \quad (50)$$
$$C_p = k^2 l_p C_s \quad (51)$$

$$\gamma(\eta, \mu) = 8 \int_{-\infty}^{\infty} z^{-p} \sin^2 (\zeta \cdot \mu/2) \sin^2 (\zeta \cdot \eta/2) d\zeta/(2\pi) \quad (52)$$
$$= D(\eta) + D(\mu) - \frac{1}{2} D(\eta + \mu) - \frac{1}{2} D(\eta - \mu). \quad (53)$$

$$D(y) = 2(R(0) - R(y)). \quad (54)$$

The one-dimensional autocorrelation function has the same form as the 3D
model:

$$R(y) = \frac{C_p}{\sqrt{\pi \Gamma(p/2)}} (q_0 y/2)^{(p-1)/2} K_{(p-1)/2} (q_0 y) / q_0^{p-1} \quad (55)$$

$$\varphi(q) = C_p (q_0^2 + q^2)^{-p/2} \quad (56)$$

Following the same development

$$R(0) = \frac{C_p \Gamma((p - 1)/2)}{2 \sqrt{\pi \Gamma(p/2)} q_0^{p-1}}, \quad (57)$$

and

$$D(y) = 2(R(0) - R(y))$$

$$= \frac{C_p \Gamma((p - 1)/2)}{\sqrt{\pi \Gamma(p/2)}} \left(1 - |q_0 y/2|^{(p-1)/2} K_{(p-1)/2} (q_0 y) / \Gamma((p - 1)/2) / q_0^{p-1}\right). \quad (58)$$
To evaluate the limit as \( q_0 \to 0 \) L’Hopital’s rule is applied to \( D(y) \):

\[
D^{(1)}(y) = \frac{C_p}{\sqrt{\pi} \Gamma(p/2)} \left( \frac{y \left| q_0 y/2 \right|^{(p-1)/2} K_{(p-3)/2} \left( q_0 y \right)}{(p-1) q_0^{p-2}} \right) \tag{59}
\]

\[
= \frac{C_p}{\sqrt{\pi} \Gamma(p/2)} \left( \frac{y \left| q_0 y/2 \right|^{(p-3)/2} K_{(p-3)/2} \left( q_0 y \right)}{(p-1) q_0^{p-2}} \right) \tag{60}
\]

With \( 1 < p < 3 \),

\[
\lim_{q_0 \to 0} D(y) = \lim_{q_0 \to 0} D^{(1)}(y)
\]

\[
= \frac{C_p \Gamma((3-p)/2)}{\sqrt{\pi} \Gamma(p/2) (p-1) 2^{p-2}} y^{|p-1|}. \tag{61}
\]

A parallel development will show that

\[
\gamma(\eta, \mu) = C(p) h(\eta, \mu) \tag{62}
\]

where

\[
C_p(p) = \frac{C_p \Gamma((3-p)/2)}{\sqrt{\pi} \Gamma(p/2) (p-1) 2^{p-1}}. \tag{63}
\]

The scalar forms are obtained from the vector forms by replacing the vector variables with their one-dimensional scalar equivalents.

### 4 Limiting Forms

#### 4.1 Weak Scatter \( U \to 0 \)

With \( \gamma(\eta, \mu) \) specified it is still necessary to use numerical integration to compute \( \Phi_I(\kappa;U, \rho_F) \) and \( SI \). However, it can be shown by direct computation that

\[
\lim_{U \to 0} \Phi_I(\kappa;U, \rho_F) = 4C_p \rho_F(\kappa) \sin^2 \left( \kappa^2 \rho_F^2 / 2 \right). \tag{64}
\]

For the three-dimensional power-law

\[
SI^2 = 4C_p \int \kappa^{-p} \sin^2 \left( \kappa^2 \rho_F^2 / 2 \right) \frac{d\kappa}{(2\pi)^2}
\]

\[
= \frac{C_p \rho_F^{p-2} - 2^{-p+1/2}}{\pi} 2 \int_0^{\infty} \mu^{-p+2} \sin^2 \left( \mu^2 \right) d\mu \tag{65}
\]

With the substitution \( p1 = 2\nu + 1 \), this is the same as the second line of (3.88) in Chapter 3. From Gradshyteyn and Ryzhik (3.861),

\[
2 \int_0^{\infty} \mu^{-p+1} \sin^2 \left( \mu^2 \right) d\mu = \int_0^{\infty} y^{-p+1/2} \sin^2 \left( \frac{\mu^2}{2} \right) dy
\]

\[
= -\frac{\Gamma(1-p1/2) \cos \left( \frac{(2-p1)\pi}{4} \right)}{2^{2-p1/2}} \text{ for } 2 < p1 < 6 \tag{66}
\]
The alternative formula

\[ 2 \int_0^\infty \mu^{-p_1 + 1} \sin^2 (\mu^2) \, d\mu = \sqrt{\pi} \frac{\Gamma ((6 - p_1) / 4)}{\Gamma (p_1/4) (p_1 - 2)} \text{ for } 2 < p_1 < 6, \tag{67} \]

came originally from mathematica, courtesy of Raul Martinez. Using the second form, it follows that

\[ S I^2 = \frac{C_{p_F}}{\sqrt{\pi 2^{p_1 + 1/2 - 1}}} \frac{\Gamma ((6 - p_1) / 4)}{\Gamma (p_1/4) (p_1 - 2)} \text{ for } 2 < p_1 < 6, \tag{68} \]

which agrees with (3.88) in Chapter 3 [1] when the substitution for \( p_1 \) is made.

### 4.1.1 Two-Dimensional Model

With the same manipulations for the two-dimensional model

\[ S I^2 = 4 C_{p_F}^{p_1 - 1} \int_{-\infty}^{\infty} \mu^{-p} \sin^2 (\mu^2/2) \frac{d\mu}{2\pi} \]

\[ = \frac{2 C_{p_F}^{p_1 - 1}}{\pi 2^{(p-1)/2}} 2 \int_0^{\infty} \xi^{-p} \sin^2 (\xi^2) \, d\xi \]

\[ = \frac{C_{p_F}^{p_1 - 1}}{\sqrt{\pi} 2^{(p-1)/2}} \frac{\Gamma ((5 - p))}{\Gamma ((p + 1)/4) (p - 1)} \text{ for } 1 < p < 5. \]

This can be derived from the three-dimensional form by replacing \( p_1 \) by \( p_1 + 1 \) and multiplying by 2. The doubling comes from the the difference between cylindrical coordinates and assuming no variation along the second axis.

### 4.2 Strong Scatter \( U \to \infty \)

With a redefinition of \( U \) for the single power-law model,

\[ \Phi_T(\mu/\rho_F)/\rho_F = \int \int \left\{ \exp \left\{ -U^{p_1 - 2} h(\eta, \kappa) \right\} - 1 \right\} \exp \left\{ -i\mu \cdot \eta \right\} \, d\eta. \tag{69} \]

where

\[ U^{p_1 - 2} = U^{p_1 - 2} C(p), \tag{70} \]

and

\[ h(\eta, \mu) = 2 |\eta|^{p_1 - 2} + 2 |\mu|^{p_1 - 2} - |\eta + \mu|^{p_1 - 2} - |\eta - \mu|^{p_1 - 2} \text{ for } p_1 \neq 4. \tag{71} \]

For large \( U \), the contributions to \( \Phi_T(\mu/\rho_F)/\rho_F \) must come from small values of \( h(\eta, \mu) \), which suggests a Taylor series expansion on the \( \eta \) variable

\[ h(\eta, \mu q) = 2 |\eta|^{p_1 - 2} + 2 q^{p_1 - 2} \left\{ 1 - |\mu + \eta q|^{p_1 - 2} / 2 - |\mu - \eta q|^{p_1 - 2} / 2 \right\}. \tag{72} \]
The symmetric difference terms in parentheses must be expanded to second order:

\[ 1 - |u + \eta/q|^{p_1-2}/2 - |u - \eta/q|^{p_1-2}/2 = -(p_1/2 - 1) \eta^2/q^2 \]

and

\[ 2q^{p_1-2} \left( 1 - |u + \eta/q|^{p_1-2}/2 - |u - \eta/q|^{p_1-2}/2 \right) \simeq -(p_1 - 2) \eta^2 q^{p_1-4} \]

\[ - (p_1 - 2) (p_1 - 4)/2 \left( 2(u \cdot \eta)^2 q^{p_1-4} + \eta^4 q^{p_1-6} \right) \]

which checks with (32) in Rino [3] when substitution for \( p_1 \) is made.

For \( 2 < p_1 < 4 \), only the first term in (75) survives. It follows that

\[ h(\eta, uq) \simeq 2|\eta|^{p_1-2} - \frac{(p_1 - 2) (\eta^2 + (p_1 - 4) (u \cdot \eta)^2)}{q^{4-p_1}}, \]

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which checks with (32) in Rino [3] when substitution for \( p_1 \) is made.
\[
\Phi_1(uq / \rho_F) / \rho_F = \frac{\pi q^{4-p_1}}{\sqrt{(p_1 - 3)(p_1 - 2)} 2U^{p_1-2}} \exp \left\{ - \frac{q^{6-p_1}}{4(p_1 - 2)(p_1 - 3) 2U^{p_1-2}} \right\}
\]  
(80)

The corresponding result (37) in Rino [3] does not have the factor-of-two multiplying \( U^{p_1-2} \). The following results were used:

\[
\int \exp \left\{ - \frac{(p_1 - 2) 2U^{p_1-2} - q^{4-p_1} \eta_1^2}{(p_1 - 2) 2U^{p_1-2}} \right\} d\eta_1 = \sqrt{\frac{\pi q^{4-p_1}}{(p_1 - 2) 2U^{p_1-2}}} \]  
(81)

\[
\int \exp \left\{ - \frac{(p_1 - 2)(p_1 - 3) 2U^{p_1-2} - q^{4-p_1} \eta_1^2}{(p_1 - 2)(p_1 - 3) 2U^{p_1-2}} \right\} \exp \{ -iq\eta_1 \} d\eta_1 = \sqrt{\frac{\pi q^{4-p_1}}{4(p_1 - 2)(p_1 - 3) 2U^{p_1-2}}} \]  
(82)

The saturation \( SI \) value for \( p_1 > 4 \) can be computed directly:

\[
SI^2 = \int_0^\infty \frac{q^{5-p_1}}{2\sqrt{(p_1 - 3)(p_1 - 2)} 2U^{p_1-2}} \exp \left\{ - \frac{q^{6-p_1}}{4(p_1 - 2)(p_1 - 3) 2U^{p_1-2}} \right\} dq
\]

\[
= 2\sqrt{(p_1 - 3)} \int_0^\infty \exp \left\{ - \frac{q^{6-p_1}}{4(p_1 - 2)(p_1 - 3) 2U^{p_1-2}} \right\} d \frac{q^{6-p_1}}{4(p_1 - 2)(p_1 - 3) 2U^{p_1-2}}
\]

\[
= 2\sqrt{(p_1 - 3)} \frac{6-p_1}{6-p_1}
\]  
(83)

This result is a factor of 2 smaller than (39) in Rino [3].

4.2.1 Two-Dimensional

Following a parallel development

\[
\Phi_1(\kappa U, \rho_F) = \int_{-\infty}^{\infty} \exp \left\{ -U^{p-1} \gamma (\alpha / \rho_F, \kappa \rho_F) \right\} - 1 \exp \{ -i\kappa \alpha \} d\alpha.
\]  
(84)

where

\[
\gamma (\eta, \mu) = C(p) h (\eta, \mu)
\]  
(85)

and

\[
h (\eta, \mu) = 2|\eta|^{p-1} + 2|\mu|^{p-1} - |\eta + \mu|^{p-1} - |\eta - \mu|^{p-1} \text{ for } p \neq 3.
\]  
(86)

For large \( U \),

\[
h (\eta, q) \approx 2|\eta|^{p-1} - \frac{(p - 1)(p - 2)\eta^2}{q^{3-p}}
\]  
(87)

For \( 1 < p < 3 \)
\[ \Phi_I(q/F)/\rho_F = \int [\exp \left\{ -2U^{p-1} |\eta|^{p_1-2} \right\} - 1] \exp \{ -i\eta \} \, d\eta \] (88)

and SI ultimately converges to unity. For \(3 < p < 5\)

\[ \Phi_I(q/F)/\rho_F = \int \exp \left\{ -\frac{(p-1)(p-2)2U^{p-1}}{q^{3-p}} \eta^2 \right\} \exp \{ -i\eta \} \, d\eta \\
= \sqrt{\frac{\pi q^{3-p}}{(p-1)(p-2)2U^{p-1}}} \exp \left\{ -\frac{q^{5-p}}{4(p-1)(p-2)2U^{p-1}} \right\} \] (89)

Similarly,

\[ SI^2 = 2\sqrt{\pi} \int \sqrt{\frac{q^{3-p}}{(p-1)(p-2)2U^{p-1}}} \exp \left\{ -\frac{q^{5-p}}{4(p-1)(p-2)2U^{p-1}} \right\} \, dq \\
= \frac{1}{\sqrt{\pi}} \int \exp \left\{ -\frac{q^{5-p}}{(p-1)(p-2)2U^{p-1}} \right\} \, dq \\
= \frac{1}{5-p} \frac{1}{\sqrt{\pi}} \int \exp \left\{ -u^2 \right\} \, du = \frac{1}{5-p} \] (90)

This result is also differs by a factor-or-two from (10) in Rino [4].
References


