Computation of Intensity Spectra for Unbounded power law phase scree.

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August 20, 2013

Abstract

This note is a recalculation of unbounded power-law phase screen intensity spectra. The analytic results resolve a long standing disparity and provide test cases for numerical analysis procedures that are being used to extend the analysis to two-component power-law spectra.

1 Introduction

The interaction of electromagnetic waves with weakly inhomogeneous structure is characterized by the forward propagation equation (FPE). The formal solution to the FPE is a complex field $\psi(k; x, \varsigma)$ where $k = 2\pi n_0 f/c$ is the wavenumber in a reference background medium with refractive index n_0, frequency f, and c is the vacuum velocity of light. The coordinate x is the propagation reference direction with ς the normal plane coordinate. The field intensity is

$$I(k; x, \varsigma) = |\psi(k; x, \varsigma)|^2 . \tag{1}$$

The statistics of $\psi(k; x, \varsigma)$ are characterized by a hierarchy of differential equations, one for each moment of the form[1]

$$\Gamma_{nm}(x; \zeta_1, \cdots \zeta_N; \xi_1, \cdots \xi_M) = \left\langle \prod_{n=1}^{N} \psi(x, \zeta_n) \prod_{m=1}^{M} \psi^*(x, \xi_m) \right\rangle . \tag{2}$$

A discussion of the specific form of the moment equations and their relation to other published results are discussed in

`http://chuckrino.com/wordpress/wp-content/uploads/2011/05/BookNotesCh32.pdf`

2 Phase Screen Model

The phase-screen model assigns the perturbation to an equivalent phase screen formed by a path integral through the structured region. Free-space propagation proceeds from the center of the phase screen to the observer. The average

[1] See (3.50) and (3.51) in Chapter 3 of Rino [1].

intensity structure is characterized by the fourth-order moment $\Gamma_{22}(x;\alpha_1,\alpha_2)$. In free space

$$\frac{\partial\Gamma_{22}(x;\alpha_1,\alpha_2)}{\partial x} = -\frac{i}{k}\nabla_{\alpha_1}\cdot\nabla_{\alpha_2}\Gamma_{22}(x;\alpha_1,\alpha_2) - k^2 g\left(\alpha_1,\alpha_2\right)\Gamma_{22}(x;\alpha_1,\alpha_2), \tag{3}$$

where

$$g\left(\alpha_1,\alpha_2\right) = D_{\delta\overline{n}}(\alpha_1) + D_{\delta\overline{n}}(\alpha_2) - D_{\delta\overline{n}}(\alpha_1+\alpha_2)/2 - D_{\delta\overline{n}}(\alpha_1-\alpha_2)/2 \tag{4}$$

$$= 8k^2\iint \Phi_{\delta n}(0,\kappa)\sin^2\left(\kappa\cdot\alpha_{\mathbf{1}}/2\right)\sin^2\left(\kappa\cdot\alpha_{\mathbf{2}}/2\right)d\kappa/(2\pi)^2. \tag{5}$$

The the integral for in (5) follows by direct computation from the definition

$$D_{\delta\overline{n}}(\alpha) = 2\left(R_{\delta\overline{n}}\left(\mathbf{0}\right) - R_{\delta\overline{n}}\left(\alpha\right)\right) = 2\iint\left(1-\exp\{i\kappa\cdot\alpha\}\right)\Phi_{\delta n}(0,\kappa)d\kappa/(2\pi)^2. \tag{6}$$

The overbar notation $\delta\overline{n}$ is a reminder of the implicit path integration, which reduces the dimensionality of the structure characterization from three to two.

2.1 Fourth Order Moment Solution

Consider the generalized Fourier transformation relation

$$\widehat{\Gamma}_{22}(x,\kappa_1,\kappa_2) = \iint\iint\Gamma_{22}(x,\alpha_1,\alpha_2)\exp\left\{-i\left(\kappa_1\cdot\alpha_1+\kappa_2\cdot\alpha_2\right)\right\}d\alpha_1 d\alpha_2. \tag{7}$$

In the absence of structure $g\left(\alpha_1,\alpha_2\right)=0$, whereby (3) reduces to

$$\frac{\partial\widehat{\Gamma}_{22}(x,\kappa_1,\kappa_2)}{\partial x} = \frac{i}{k}\kappa_1\cdot\kappa_2\widehat{\Gamma}_{22}(x,\kappa_1,\kappa_2). \tag{8}$$

In the absence of diffraction $\nabla_{\alpha_1}\cdot\nabla_{\alpha_2}\Gamma_{22}(x,\alpha_1,\alpha_2)=0$, (3) reduces to

$$\frac{\partial\Gamma_{22}(x,\alpha_1,\alpha_2)}{\partial x} = -k^2 g\left(\alpha_1,\alpha_2\right)\Gamma_{22}(x,\alpha_1,\alpha_2). \tag{9}$$

A formal solution to the fourth-order moment equation can be constructed as follows: Assume that $\Gamma_{22}(x,\alpha_1,\alpha_2)$ is known. First apply the structure contribution,

$$\Gamma_{22}^{1/2}(x,\alpha_1,\alpha_2) = \Gamma_{22}(x,\alpha_1,\alpha_2)\exp\left\{-k^2 g\left(\alpha_1,\alpha_2\right)\Delta x\right\}. \tag{10}$$

To propagate $\Gamma_{22}^{1/2}(x,\alpha_1,\alpha_2)$ the generalized Fourier transform is computed. The propagation operator implied by the solution to (8) is then applied:

$$\widehat{\Gamma}_{22}(x+\Delta x,\kappa_1,\kappa_2) = \iint\left[\iint \Gamma_{22}^{1/2}(x,\alpha_1',\alpha_2') \right.$$
$$\times \exp\{-i\left(\kappa_1\cdot\alpha_1' + \kappa_2\cdot\alpha_2'\right)\}d\alpha_1' d\alpha_2'\Big]$$
$$\times \exp\{i\kappa_1\cdot\kappa_2\Delta x/k\}. \tag{11}$$

The new field at $x+\Delta x$ is reconstructed by computing the inverse Fourier transformation

$$\Gamma_{22}(x+\Delta x,\alpha_1,\alpha_2) = \iint \widehat{\Gamma}_{22}(x+\Delta x,\kappa_1,\kappa_2)$$
$$\times \exp\{i\left(\kappa_1\cdot\alpha_1+\kappa_2\cdot\alpha_2\right)\}\frac{d\kappa_1}{(2\pi)^2}\frac{d\kappa_2}{(2\pi)^2}. \tag{12}$$

2.2 Intensity SDF

In the transformed coordinate system the intensity correlation function is obtained from $\Gamma_{22}(x;\alpha_1,\alpha_2)$ by setting $\alpha_2 = 0$:

$$\langle I\left(k;x,\varsigma\right) I\left(k;x,\varsigma'\right)\rangle = \Gamma_{22}(x;\alpha_1,\mathbf{0}). \tag{13}$$

The intensity SDF is the Fourier transformation of $\Gamma_{22}(x;\alpha_1,\mathbf{0})$, but forward propagation of the intensity spectrum involves both the α_1 and α_2 dependencies. One can show that

$$\Phi_I(\kappa) = \iint \Gamma_{22}(x_0,\alpha,\kappa\rho_F^2)\exp\{-i\kappa\cdot\alpha\}\,d\alpha,$$

where

$$\rho_F = \sqrt{x/k}. \tag{14}$$

The split-step solution is simplified significantly if the structured region is replaced by a single *equivalent* phase screen. At the exit plane of the phase screen

$$\Gamma_{22}(l_p,\alpha_1,\alpha_2) = \exp\left\{-l_p k^2 g\left(\alpha_1,\alpha_1\right)\right\}. \tag{15}$$

The subsequent propagation of the intensity SDF in computed as

$$\Phi_I(\kappa) = \iint \left[\exp\left\{-l_p k^2 g\left(\alpha,\kappa\rho_F^2\right)\right\}-1\right]\exp\{-i\kappa\cdot\alpha\}\,d\alpha. \tag{16}$$

Subtracting 1 from the integrand effectively removes the mean intensity, which is conserved by the FPE. It follows that

$$SI^2 = < I^2 > -1 \tag{17}$$
$$= \iint \Phi_I(\kappa)d\kappa/\left(2\pi\right)^2 \tag{18}$$

3 Structure Model

To accommodate oblique propagation the FPE is rewritten in a continuously displaced coordinate system (CDC). As discussed in Chapter 4 of Rino [1], geometric dependence of the path integral in the CDC system can be accommodated by replacing $\Phi_{\delta n}(0,\kappa)$ with $\Phi_{\delta n}(\tan\theta\widehat{\mathbf{a}}_{k_T}\cdot\kappa,\kappa)$. The effect is to replace κ^2 with a quadratic form. A general inverse-power-law SDF form is constructed as

$$\Phi_{\delta n}(q) = C_s\varphi(q), \tag{19}$$

where

$$\varphi(q) = \begin{cases} q^{-p_1} & q < q_0 \\ q_0^{p_2-p_1}q^{-p_2} & q > q_0 \end{cases}. \tag{20}$$

Note that with $p_1 = 0$, the transition wavenumber, q_0, becomes an outer scale. With $p_2 = p_1$, the spectrum is formally an unmodified power law. With $p_2 > p_1$, the q_0 weighting ensures continuity at the break point.

With some substitutions and manipulations

$$\begin{aligned} l_pk^2g\left(\alpha,\kappa\rho_F^2\right) &= 8C_p\rho_F^{-2}\iint\varphi(\kappa')\sin^2\left(\kappa'\rho_F\cdot\left(\alpha/\rho_F\right)/2\right) \\ &\times\sin^2\left(\kappa'\rho_F\cdot\kappa\rho_F/2\right)d\left(\kappa'\rho_F\right)/(2\pi)^2 \\ &= 8C_p\rho_F^{p1-2}\iint\varphi(\varkappa/\rho_F)\sin^2\left(\varkappa\cdot\left(\alpha/\rho_F\right)/2\right) \\ &\times\sin^2\left(\varkappa'\cdot\kappa\rho_F/2\right)d\varkappa'/(2\pi)^2 \end{aligned} \tag{21}$$

Let

$$\gamma\left(\eta,\mu\right) = 8\iint\begin{cases} \varkappa^{-p_1} & \varkappa < q_0 \\ \varkappa_0^{p_2-p_1}\varkappa^{-p_2} & \varkappa > q_0 \end{cases}\sin^2\left(\varkappa\cdot\eta/2\right)\sin^2\left(\varkappa\cdot\mu/2\right)d\varkappa/(2\pi)^2. \tag{22}$$

It follows that

$$l_pk^2g\left(\alpha,\kappa\rho_F^2\right) = U^{p_1-2}\gamma\left(\alpha/\rho_F,\kappa\rho_F\right), \tag{23}$$

where

$$U = C_p^{\frac{1}{p_1-2}}\rho_F \tag{24}$$

and

$$C_p = k^2l_pC_s. \tag{25}$$

$$\Phi_I(\kappa) = \rho_F\iint\left[\exp\left\{-U^{p_1-2}\gamma\left(\alpha/\rho_F,\kappa\rho_F\right)\right\}-1\right]\exp\left\{-i\kappa\rho_F\cdot\alpha/\rho_F\right\}d\alpha/\rho_F \tag{26}$$

or

$$\Phi_I(\mu/\rho_{\mathbf{F}})/\rho_F = \iint\left[\exp\left\{-U^{p_1-2}\gamma\left(\eta,\mu\right)\right\}-1\right]\exp\left\{-i\mu\cdot\eta\right\}d\eta. \tag{27}$$

3.1 Single Power Law

To introduce the single power-law model consider the constrained isotropic form:

$$R\left(y\right)=\frac{C_{p}}{2\pi\Gamma(p_{1}/2)}\left(q_{0}y/2\right)^{p_{1}/2-1}K_{p_{1}/2-1}\left(q_{0}y\right)/q_{0}^{p_{1}-2} \tag{28}$$

$$\varphi(q)=C_{p}\left(q_{0}^{2}+q^{2}\right)^{-p_{1}/2} \tag{29}$$

The unconstrained power-law form is obtained by taking the limit as the outer scale wavenumber $q_0 \to 0$. The small argument formula is used to evaluate the modified Bessel functions:

$$\lim_{x\longrightarrow0}K_{\gamma}\left(x\right)=\frac{1}{2}\Gamma(|\gamma|)\left(x/2\right)^{-|\gamma|}\quad\text{for }\gamma\neq0. \tag{30}$$

The absolute values are introduced to ensure that the relation $K_{\gamma}\left(x\right)=K_{-\gamma}\left(x\right)$ is preserved. It follows that

$$\lim_{x\longrightarrow0}\left(x/2\right)^{\gamma}K_{\gamma}\left(x\right)=\frac{1}{2}\Gamma(|\gamma|)\left\{\begin{array}{cc}1 & \text{for }\gamma>0\\ \left(x/2\right)^{-2|\gamma|} & \text{for }\gamma<0\end{array}\right. . \tag{31}$$

The sign of the power-law index deterines the limiting form of the product as $x \to 0$.

Using the limiting foumula with $p > 2$, it follows from (28) that

$$R(0)=\frac{C_{p}\Gamma(p_{1}/2-1)}{4\pi\Gamma(p_{1}/2)q_{0}^{p_{1}-2}}, \tag{32}$$

and

$$\begin{aligned}D(y)&=2(R(0)-R(y))\\ &\frac{C_{p}\Gamma(p_{1}/2-1)}{2\pi\Gamma(p_{1}/2)}\left(\frac{1-2\left(q_{0}y/2\right)^{p_{1}/2-1}K_{p_{1}/2-1}\left(q_{0}y\right)/\Gamma(p_{1}/2-1)}{q_{0}^{p_{1}-2}}\right).\end{aligned} \tag{33}$$

Following L´Hospital's rule the indeterminate form $\lim_{q_0\longrightarrow0}D(y)\sim0/0$ can be evaluated by taking the limit of the ratio of the derivatives of the numerator and the denominator. Derivatives of the K-type Bessel function terms follow from the relations

$$\frac{d}{dx}\left\{x^{\pm|\gamma|}K_{\gamma}\left(x\right)\right\}=-x\left\{x^{\mp|\gamma|-1}K_{|\gamma|\mp1}\left(x\right)\right\}. \tag{34}$$

For $2 < p_1 < 4$,

$$D^{(1)}(y)=\frac{-2C_{p}}{2\pi\Gamma(p_{1}/2)}\left(\frac{\frac{d}{dq_{0}}\left(q_{0}y/2\right)^{p_{1}/2-1}K_{p_{1}/2-1}\left(q_{0}y\right)}{\frac{d}{dq_{0}}q_{0}^{p_{1}-2}}\right). \tag{35}$$

From (34) with the upper sign

$$\frac{d}{dq_0}\left(q_0 y/2\right)^{p_1/2-1} K_{p_1/2-1}\left(q_0 y\right) = -y\left(q_0 y/2\right)\left\{\left(q_0 y/2\right)^{p_1/2-2} K_{p_1/2-2}\left(q_0 y\right)\right\}. \tag{36}$$

Thus,

$$D^{(1)}(y) = \frac{2C_p}{2\pi\Gamma(p_1/2)}\left(\frac{y\left(q_0 y/2\right)\left(q_0 y/2\right)^{p_1/2-2} K_{p_1/2-2}\left(q_0 y\right)}{\left(p_1-2\right) q_0^{p_1-3}}\right), \tag{37}$$

whereby

$$\begin{aligned}\lim_{q_0\to 0} D(y) &= \lim_{q_0\to 0} D^{(1)}(y)\\ &= \frac{C_p\Gamma(2-p_1/2)}{2\pi\Gamma(p_1/2)\left(p_1-2\right)2^{p_1-3}} y^{p_1-2}.\end{aligned} \tag{38}$$

From (6), it follows that the structure function is not defined for an unconstrained power-law with $p_1 \geq 4$. However, it follows from (5) that $g\left(\alpha_1, \alpha_2\right)$ is defined for $2 < p_1 < 6$.

To pursue this, note first that from (34) with $p_1 > 4$

$$\lim_{q_0\to 0} D^{(1)}(y) = \frac{2C_p}{2\pi\Gamma(p_1/2)} \lim_{q_0\to 0}\left(\frac{y\left(q_0 y/2\right)\Gamma(p_1/2-2)/2}{\left(p_1-2\right) q_0^{p_1-3}}\right), \tag{39}$$

which is indeterminate. However, L´Hospital's rule can be applied to $D^{(1)}(y)$ modified to accommodate the sign change of $p_1/2-2$ when p_1 exceeds 4:

$$\begin{aligned}D^{(2)}(y) &= \frac{2C_p}{2\pi\Gamma(p_1/2)}\left(\frac{\left(y^2/2\right)\frac{d}{dq_0} q_0\left(q_0 y/2\right)^{(p_1/2-2)} K_{p_1/2-2}\left(q_0 y\right)}{\frac{d}{dq_0}\left(p_1-2\right) q_0^{p_1-3}}\right)\\ &= \frac{2C_p}{2\pi\Gamma(p_1/2)}\\ &\times\left(\frac{\left(y^2/2\right)\left(q_0 y/2\right)^{-(p_1/2-2)} K_{p_1/2-2}\left(q_0 y\right) + \left(q_0 y^2/2\right)\frac{d}{dq_0}\left(q_0 y/2\right)^{(p_1/2-2)} K_{p_1/2-2}\left(q_0 y\right)}{\left(p_1-2\right)\left(p_1-3\right) q_0^{p_1-4}}\right)\\ &= \frac{2C_p}{2\pi\Gamma(p_1/2)}\\ &\times\left(\frac{\left(y^2/2\right)\left(q_0 y/2\right)^{-(p_1/2-2)} K_{p_1/2-2}\left(q_0 y\right) - \left(q_0 y^2/2\right) y\left(q_0 y/2\right)^{(p_1/2-2)} K_{p_1/2-3}\left(q_0 y\right)}{\left(p_1-2\right)\left(p_1-3\right) q_0^{p_1-4}}\right)\end{aligned} \tag{40}$$

Taking the limit as $q_0 \to 0$,

$$\lim_{q_0 \to 0} D^{(2)}(y) = \frac{2C_p}{2\pi\Gamma(p_1/2)}$$
$$\times \lim_{q_0 \to 0} \left(\frac{\left(y^2/2\right)\left(q_0 y/2\right)^{-(p_1/2-2)} K_{p_1/2-2}\left(q_0 y\right) - \left(q_0 y^2/2\right) y \left(q_0 y/2\right)^{(p_1/2-2)} K_{p_1/2-3}\left(q_0 y\right)}{\left(p_1 - 2\right)\left(p_1 - 3\right) q_0^{p_1-4}} \right)$$
$$= \frac{C_p}{2\pi\Gamma(p_1/2)\left(p_1 - 2\right)}$$
$$\times \lim_{q_0 \to 0} \left(\frac{\left(y^2/2\right)\left(q_0 y/2\right)^{(p_1-4)} \Gamma(p_1/2 - 1) - \left(q_0 y^2/2\right) y \left(q_0 y/2\right)^{(p_1-5)} \Gamma(3 - p_1/2)}{\left(p_1 - 2\right)\left(p_1 - 3\right) q_0^{p_1-4}} \right)$$
$$= \frac{C_p y^{(p_1-2)}}{2\pi\Gamma(p_1/2)\left(p_1 - 2\right) 2^{(p_1-3)}} \left(\frac{\Gamma(p_1/2 - 1) - \Gamma(3 - p_1/2)/2}{\left(p_1 - 3\right)} \right). \tag{41}$$

Using the relation

$$\Gamma(3 - p_1/2) = \Gamma(2 - p_1/2 + 1) = (2 - p_1/2)\,\Gamma(2 - p_1/2), \tag{42}$$

it follows that

$$\begin{aligned} \Gamma(p_1/2 - 1) - \Gamma(3 - p_1/2)/2 &= \Gamma(p_1/2 - 1) - (2 - p_1/2)\,\Gamma(2 - p_1/2)/2 \\ &= \Gamma(p_1/2 - 1)(1 - (4 - p_1) \\ &= -(p1 - 3)\Gamma(p_1/2 - 1). \end{aligned} \tag{43}$$

Making the appropiate substitutions

$$\lim_{q_0 \to 0} D(y) = \frac{-C_p y^{(p_1-2)} \Gamma(p_1/2 - 1)}{2\pi\Gamma(p_1/2)\left(p_1 - 2\right) 2^{(p_1-3)}}. \tag{44}$$

Whereas, (44) has no meaning as a stand-alone computation, the application of L' Hopital's rule to $\gamma\left(\eta, \mu\right)$ term by term yields the following result:

$$\gamma\left(\eta, \mu\right) = C(p) h\left(\eta, \mu\right) \tag{45}$$

where

$$C(p) = \frac{C_p \Gamma(p_1/2 - 1)}{2\pi\Gamma(p_1/2)\left(p_1 - 2\right) 2^{(p_1-3)}} \tag{46}$$

and

$$h\left(\eta, \mu\right) = 2\left|\eta\right|^{p_1-2} + 2\left|\mu\right|^{p_1-2} - \left|\eta + \mu\right|^{p_1-2} - \left|\eta - \mu\right|^{p_1-2} \text{ for } p_1 \neq 4. \tag{47}$$

The sign change of $\Gamma(p_1/2 - 1)$ when p_1 exceeds 4 is the only formal change in the result.

One can use the limiting form $K_0\left(x\right) \to -\ln x$ to obtain a result at $p_1 = 4$, but limit is not consistent when evaluated from the result for $p_1 > 4$. Aside from $p_1 = 4$, the result is identical to the result derived by Rumsey [2], who used integration by parts of the isotropic form

$$D_{\delta\overline{n}}(y) = \int_0^\infty (1 - J_0(\kappa y))\, \kappa^{-(p-1)} d\kappa/(2\pi). \tag{48}$$

The correct manipulation of the limiting forms of derivatives of modified Bessel functions resolves a long standing disparity noted in [3].

3.2 Two-Dimensional Models

The two-dimensional counterpart to the single power-law model is summarized as follows:

$$\Phi_I(\kappa; U, \rho_F) = \int_{-\infty}^{\infty} \left(\exp\left\{-U^{p-1}\gamma(\alpha/\rho_F, \kappa\rho_F)\right\} - 1\right) \exp\{-i\kappa\alpha\}\, d\alpha. \tag{49}$$

$$U = C_p^{\frac{1}{p-1}} \rho_F \tag{50}$$

$$C_p = k^2 l_p C_s \tag{51}$$

$$\gamma(\eta, \mu) = 8 \int_{-\infty}^{\infty} \varkappa^{-p} \sin^2(\varkappa \cdot \mu/2) \sin^2(\varkappa \cdot \eta/2)\, d\varkappa/(2\pi) \tag{52}$$

$$= D(\eta) + D(\mu) - \frac{1}{2}D(\eta + \mu) - \frac{1}{2}D(\eta - \mu). \tag{53}$$

$$D(y) = 2(R(0) - R(y)). \tag{54}$$

The one-dimensional autocorrelation function has the same form as the 3D model:

$$R(y) = \frac{C_p}{\sqrt{\pi}\Gamma(p/2)} (q_0 y/2)^{(p-1)/2} K_{(p-1)/2}(q_0 y)/q_0^{p-1} \tag{55}$$

$$\varphi(q) = C_p \left(q_0^2 + q^2\right)^{-p/2} \tag{56}$$

Following the same development

$$R(0) = \frac{C_p \Gamma((p-1)/2)}{2\sqrt{\pi}\Gamma(p/2) q_0^{p-1}}, \tag{57}$$

and

$$\begin{aligned} D(y) &= 2(R(0) - R(y)) \\ &= \frac{C_p \Gamma((p-1)/2)}{\sqrt{\pi}\Gamma(p/2)} \left(\frac{1 - |q_0 y/2|^{(p-1)/2} K_{(p-1)/2}(q_0 y)/\Gamma((p-1)/2)}{q_0^{p-1}}\right). \end{aligned} \tag{58}$$

To evaluate the limit as $q_0 \to 0$ L'hopital's rule is applied to $D(y)$:

$$D^{(1)}(y) = \frac{C_p}{\sqrt{\pi}\Gamma(p/2)} \left(\frac{y\,|q_0 y/2|^{(p-1)/2}\, K_{(p-3)/2}\,(q_0 y)}{(p-1)\, q_0^{p-2}} \right) \tag{59}$$

$$= \frac{C_p}{\sqrt{\pi}\Gamma(p/2)} \left(\frac{y\,|q_0 y|\,|q_0 y/2|^{(p-3)/2}\, K_{(p-3)/2}\,(q_0 y)}{(p-1)\, q_0^{p-2}} \right) \tag{60}$$

With $1 < p < 3$,

$$\lim_{q_0 \to \to} D(y) = \lim_{q_0 \longrightarrow 0} D^{(1)}(y)$$

$$= \frac{C_p \Gamma\left((3-p)/2\right)}{\sqrt{\pi}\Gamma(p/2)\,(p-1)\,2^{p-2}}\, |y|^{(p-1)}\,. \tag{61}$$

A parallel development will show that

$$\gamma\,(\eta, \mu) = C(p) h\,(\eta, \mu) \tag{62}$$

where

$$C_p(p) = \frac{C_p\;\Gamma\left((3-p)/2\right)}{\sqrt{\pi}\Gamma(p/2)\,(p-1)\,2^{p-1}}. \tag{63}$$

The scalar forms are obtained from the vector forms by replacing the vector variables with their one-dimensional scalar equivalents.

4 Limiting Forms

4.1 Weak Scatter $U \to 0$

With $\gamma\,(\eta, \mu)$ specified it is still necessary to use numerical integration to compute $\Phi_I(\kappa;U, \rho_F)$ and SI. However, it can be shown by direct computation that

$$\lim_{U \to 0} \Phi_I(\kappa;U, \rho_F) = 4C_p \varphi(\kappa) \sin^2\left(\kappa^2 \rho_F^2/2\right). \tag{64}$$

For the three-dimensional power-law

$$SI^2 = 4C_p \iint \kappa^{-p1} \sin^2\left(\kappa^2 \rho_F^2/2\right) \frac{d\kappa}{(2\pi)^2}$$

$$= \frac{C_p \rho_F^{p_1-2} 2^{-p1/2+1}}{\pi} 2 \int_0^\infty \mu^{-p1+1} \sin^2\left(\mu^2\right) d\mu \tag{65}$$

With the substitution $p1 = 2\nu + 1$, this is the same as the second line of (3.88) in Chapter 3. From Gradshteyn and Ryzhik (3.861),

$$2 \int_0^\infty \mu^{-p1+1} \sin^2\left(\mu^2\right) d\mu = \int_0^\infty y^{-p1/2} \sin^2 y dy$$

$$= -\frac{\Gamma(1 - p1/2) \cos \frac{(2-p1)\pi}{4}}{2^{2-p1/2}} \quad \text{for } 2 < p1 < 6 \tag{66}$$

The alternative formula

$$2\int_0^{\infty}\mu^{-p1+1}\sin^2\left(\mu^2\right)d\mu=\sqrt{\pi}\frac{\Gamma\left(\left(6-p1\right)/4\right)}{\Gamma\left(p1/4\right)\left(p1-2\right)}\quad\text{for } 2<p1<6, \tag{67}$$

came originally from mathematica, courtesy of Raul Martinez. Using the second form, It follows that

$$SI^2=\frac{C_p\rho_F^{p_1-2}}{\sqrt{\pi}2^{p1/2-1}}\frac{\Gamma\left(\left(6-p1\right)/4\right)}{\Gamma\left(p1/4\right)\left(p1-2\right)}\text{ for } 2<p1<6\ , \tag{68}$$

which agrees with (3.88) in Chapter 3 [1] when the substitution for $p1$ is made.

4.1.1 Two-Dimensional Model

With the same manipulations for the two-dimensional model

$$\begin{aligned}SI^2&=4C_p\rho_F^{p-1}\int_{-\infty}^{\infty}\mu^{-p}\sin^2\left(\mu^2/2\right)\frac{d\mu}{2\pi}\\&=\frac{2}{\pi}\frac{C_p\rho_F^{p-1}}{2^{(p-1)/2}}2\int_0^{\infty}\xi^{-p}\sin^2\left(\xi^2\right)d\xi\\&=\frac{C_p\rho_F^{p-1}}{\sqrt{\pi}2^{(p-1)/2}}\frac{\Gamma\left(\left(5-p\right)\right)}{\Gamma\left(\left(p+1\right)/4\right)\left(p-1\right)}\text{ for } 1<p<5.\end{aligned}$$

This can be derived from the three-dimensional form by replacing $p1$ by $p1+1$ and multiplying by 2. The doubling comes from the the difference between cylindrical coordinates and assuming no variation along the second axis.

4.2 Strong Scatter $U\to\infty$

With a redefinition of U for the single power-law model,

$$\Phi_I(\mu/\rho_{\mathbf{F}})/\rho_F=\iint[\exp\left\{-\overline{U}^{p_1-2}h\left(\eta,\kappa\right)\right\}-1]\exp\left\{-i\mu\cdot\eta\right\}d\eta. \tag{69}$$

where

$$\overline{U}^{p_1-2}=U^{p1-2}C(p), \tag{70}$$

and

$$h\left(\eta,\mu\right)=2\left|\eta\right|^{p_1-2}+2\left|\mu\right|^{p_1-2}-\left|\eta+\mu\right|^{p_1-2}-\left|\eta-\mu\right|^{p_1-2}\text{ for } p_1\neq 4. \tag{71}$$

For large U, the contributions to $\Phi_I(\mu/\rho_{\mathbf{F}})/\rho_F$ must come from small values of $h\left(\eta,\mu\right)$, which suggests a Taylor series expansion on the η variable

$$h\left(\eta,\mathbf{u}q\right)=2\left|\eta\right|^{p_1-2}+2q^{p_1-2}\left(1-\left|\mathbf{u}+\eta/q\right|^{p_1-2}/2-\left|\mathbf{u}-\eta/q\right|^{p_1-2}/2\right). \tag{72}$$

The symmetric difference terms in parentheses must be expanded to second order: $2\sqrt{\pi}$whereby

$$1-|\mathbf{u}+\eta/q|^{p_1-2}/2-|\mathbf{u}-\eta/q|^{p_1-2}/2=-(p_1/2-1)\,\eta^2/q^2 \tag{73}$$
$$-(p_1/2-1)(p_1/2-2)/2\left(4(\mathbf{u}\cdot\eta)^2/q^2+\eta^4/q^4\right),$$

and

$$2q^{p_1-2}\left(1-|\mathbf{u}+\eta/q|^{p_1-2}/2-|\mathbf{u}-\eta/q|^{p_1-2}/2\right)\simeq-(p_1-2)\,\eta^2q^{p_1-4}$$
$$-(p_1-2)(p_1-4)/2\left(2(\mathbf{u}\cdot\eta)^2q^{p_1-4}+\eta^4q^{p_1-6}\right). \tag{74}$$

It follows that

$$h(\eta,\mathbf{u}q)\qquad\simeq 2|\eta|^{p_1-2}-\frac{(p_1-2)\left(\eta^2+(p_1-4)(\mathbf{u}\cdot\eta)^2\right)}{q^{4-p_1}}, \tag{75}$$

which checks with (32) in Rino [3] when substitution for p_1 is made.

For $2<p1<4$, only the first term in (75) survives. It follows that

$$\Phi_I(\mathbf{u}q/\rho_{\mathbf{F}})/\rho_F=\iint[\exp\left\{-2\overline{U}^{p_1-2}|\eta|^{p_1-2}\right\}-1]\exp\{-i\mathbf{u}q\cdot\eta\}\,d\eta. \tag{76}$$

The integral is not amenable to further analytic simplification, but one can show that

$$SI^2=\iint\Phi_I(\mu/\rho_{\mathbf{F}})/\rho_F\frac{d\mu}{(2\pi)^2}=1. \tag{77}$$

The scintillation can exceed unity with increasing $\overline{U}$, but ultimately it will come back to the Rayleigh limit of unity.

For $4<p1<6$, the second term in (75) dominates, whereby

$$\Phi_I(\mathbf{u}q/\rho_{\mathbf{F}})/\rho_F=\iint\exp\left\{-\frac{(p_1-2)\left(\eta^2+(p_1-4)(\mathbf{u}\cdot\eta)^2\right)2\overline{U}^{p_1-2}}{q^{4-p_1}}\right\}\exp\{-i\mathbf{u}q\cdot\eta\}\,d\eta, \tag{78}$$

The singular contribution is eliminated in the large q limit. Choose the integration axes so that $\mathbf{u}\cdot\eta=\eta_y$.

$$\begin{aligned}\Phi_I(\mathbf{u}q/\rho_{\mathbf{F}})/\rho_F&=\iint\exp\left\{-\frac{(p_1-2)\left(\eta_x^2+(p_1-3)\,\eta_y^2\right)2\overline{U}^{p_1-2}}{q^{4-p_1}}\right\}\exp\{-iq\eta_y\}\,d\eta\\&=\int\exp\left\{-\frac{(p_1-2)\,2\overline{U}^{p_1-2}}{q^{4-p_1}}\eta_x^2\right\}d\eta_x\\&\times\int\exp\left\{-\frac{(p_1-2)(p_1-3)\,2\overline{U}^{p_1-2}}{q^{4-p_1}}\eta_y^2\right\}\exp\{-iq\eta_y\}\,d\eta\end{aligned} \tag{79}$$

$$\Phi_I(\mathbf{u}q/\rho_{\mathbf{F}})/\rho_F = \frac{\pi q^{4-p_1}}{\sqrt{(p_1-3)}\,(p_1-2)\,2\overline{U}^{p_1-2}} \exp\left\{-\frac{q^{6-p_1}}{4\,(p_1-2)\,(p_1-3)\,2\overline{U}^{p_1-2}}\right\} \tag{80}$$

The corresponding result (37) in Rino [3] does not have the factor-of-two multiplying $\overline{U}^{p_1-2}$. The following results were used:

$$\int \exp\left\{-\frac{(p_1-2)\,2\overline{U}^{p_1-2}}{q^{4-p_1}}\eta_x^2\right\} d\eta_x = \sqrt{\frac{\pi q^{4-p_1}}{(p_1-2)\,2\overline{U}^{p_1-2}}} \tag{81}$$

$$\begin{aligned} &\int \exp\left\{-\frac{(p_1-2)\,(p_1-3)\,2\overline{U}^{p_1-2}}{q^{4-p_1}}\eta_y^2\right\} \exp\{-iq\eta_y\}\, d\eta \\ &= \sqrt{\frac{\pi q^{4-p_1}}{(p_1-2)\,(p_1-3)\,2\overline{U}^{p_1-2}}} \exp\left\{-\frac{q^2 q^{4-p_1}}{4\,(p_1-2)\,(p_1-3)\,2\overline{U}^{p_1-2}}\right\} \end{aligned} \tag{82}$$

The saturation SI value for $p1 > 4$ can be computed directly:

$$\begin{aligned} SI^2 &= \int_0^\infty \frac{q^{5-p_1}}{2\sqrt{(p_1-3)}\,(p_1-2)\,2\overline{U}^{p_1-2}} \exp\left\{-\frac{q^{6-p_1}}{4\,(p_1-2)\,(p_1-3)\,2\overline{U}^{p_1-2}}\right\} dq \\ &= \frac{2\sqrt{(p_1-3)}}{6-p_1} \int_0^\infty \exp\left\{-\frac{q^{6-p_1}}{4\,(p_1-2)\,(p_1-3)\,2\overline{U}^{p_1-2}}\right\} d\frac{q^{6-p_1}}{4\,(p_1-2)\,(p_1-3)\,2\overline{U}^{p_1-2}} \\ &= \frac{2\sqrt{(p_1-3)}}{6-p_1} \end{aligned} \tag{83}$$

This result is a factor of 2 smaller than (39) in Rino [3].

4.2.1 Two-Dimensional

Following a parallel development

$$\Phi_I(\kappa;U,\rho_F) = \int_{-\infty}^{\infty} \left(\exp\left\{-U^{p-1}\gamma\left(\alpha/\rho_F, \kappa\rho_F\right)\right\} - 1\right) \exp\{-i\kappa\alpha\}\, d\alpha. \tag{84}$$

where

$$\gamma\left(\eta,\mu\right) = C(p)h\left(\eta,\mu\right) \tag{85}$$

and

$$h\left(\eta,\mu\right) = 2\left|\eta\right|^{p-1} + 2\left|\mu\right|^{p-1} - \left|\eta+\mu\right|^{p-1} - \left|\eta-\mu\right|^{p-1} \text{ for } p \neq 3. \tag{86}$$

For large U,

$$h\left(\eta,q\right) \simeq 2\left|\eta\right|^{p-1} - \frac{(p-1)(p-2)\eta^2}{q^{3-p}} \tag{87}$$

For $1 < p < 3$

$$\Phi_I(q/\rho_F)/\rho_F = \int [\exp\left\{-2\overline{U}^{p-1}\left|\eta\right|^{p_1-2}\right\} - 1] \exp\left\{-iq\eta\right\} d\eta \tag{88}$$

and SI ultimately converges to unity. For $3 < p < 5$

$$\begin{aligned}\Phi_I(q/\rho_F)/\rho_F &= \int \exp\left\{-\frac{(p-1)(p-2)2\overline{U}^{p-1}}{q^{3-p}}\eta^2\right\} \exp\left\{-iq\eta\right\} d\eta \\ &= \sqrt{\frac{\pi q^{3-p}}{(p-1)(p-2)2\overline{U}^{p-1}}} \exp\left\{-\frac{q^{5-p}}{4(p-1)(p-2)2\overline{U}^{p-1}}\right\}\end{aligned} \tag{89}$$

Similarly,

$$\begin{aligned}SI^2 &= 2\sqrt{\pi} \int \sqrt{\frac{q^{3-p}}{(p-1)(p-2)2\overline{U}^{p-1}}} \exp\left\{-\frac{q^{5-p}}{4p-1)(p-2)2\overline{U}^{p-1}}\right\} \frac{dq}{2\pi} \\ &= \frac{1}{\sqrt{\pi}} \int \exp\left\{-\frac{q^{5-p}}{(p-1)(p-2)2\overline{U}^{p-1}}\right\} d\frac{q^{(5-p)/2}}{\sqrt{4(p-1)(p-2)2\overline{U}^{p-1}}} \\ &= \frac{1}{5-p}\frac{1}{\sqrt{\pi}} \int \exp\left\{-u^2\right\} du = \frac{1}{5-p}\end{aligned} \tag{90}$$

This result is also differs by a factor-or-two from (10) in Rino [4].

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