# A Software Defined Phase Locked Loop

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**Abstract**

Processing low orbiting beacon satellite data has been a challenge largely because of the limited vehicle space for a VHF, UHF, L-band antenna system. The effective antenna patterns severely limit range of uniform illumination range with a common phase center. Low SNR, particularly at acquisition severely, constrains phase-locked-loop (PLL) performance. However, with post-pass processing substantially more processing can be devoted to the critical frequency tracking operation. Several schemes have been used effectively to construct a narrow band filter centered on the Doppler frequency defined over a local time segment. An analysis of these procedures was presented in a paper *Digital Signal Processing for Ionospheric Propagation Diagnostics*, by Rino, Groves, Carrano, Gunter, and Parrish.

In the paper it was hypothesized but not demonstrated that typical PLL performance was degraded for SNRs below 10 dB to the point that direct frequency estimation was a preferable alternative. This note implements a second-order PLL for direct comparison to the frequency-hypothesis tracker described in the paper.

## 1 Introduction

Phase locked loops (PLLs) have been used since the early development of radio. Critical paper collections [1] and books [2] cover the early subject matter, which includes nonlinear feedback and stochastic differential equations. Contributions have been made by information theory pioneers such as H. L. Van Trees and A. J. Viterbi. The subject matter can be intimidating, but software emulators are readily constructed for analysis and, with software defined radio, direct implementation.

An application of particular interest is processing radio signals from earth orbiting satellites. In free space the changing path between the source and receiver induces a time-varying phase change. The earth's ionosphere induces an additional phase change and, under disturbed conditions, a random complex modulation. To demodulate the signal the slowly varying carrier frequency offset must be estimated.

The following signal model captures the essential elements of the problem:

$$v(k) = \sqrt{SNR(k)}m(k)\exp\left\{i\left(\omega_0 k\Delta t + \int\limits_0^{k\Delta t}\omega(t')\,dt'\right)\right\} + \varsigma(k). \qquad (1)$$

The first term in the exponential argument is the phase progression of the constant center frequency $f_0 = \omega_0/(2\pi)$. The second term is the a phase variation represented as the integral of a slowly changing instantaneous frequency, $\omega(t)$. The signal amplitude is defined by the signal-to-noise power ratio, $SNR(k)$, which includes path and system losses as well as antenna and amplifier gains.

The component $m(k)$ represents modulation imparted at transmission. The average intensity of the modulation is constrained to unity for consistency with the $SNR$ definition. Critical sampling at $BW > 1/\Delta t$ Hz captures the modulation frequency content plus the extremes of $\omega(t)$. The term $\varsigma(k)$ is a unit variance zero mean uncorrelated complex random sequence representing receiver noise. A more complete model would include an additional complex modulation induced by propagation disturbances. The simpler model is adequate for PLL evaluation.

The first signal processing operation is estimation of the phase progression represented by the integral in the argument of the exponential. Three time scales are involved, the sample interval $\Delta t$ has already been introduced. The duration of the modulation $T$ is the interval for matched filter processing. Let $\tau_e$ represent the time scale for change in $\omega(t)$. The noise background ideally is uncorrelated over intervals greater than $\Delta t$. To the extent that $\Delta t \ll T \ll \tau_e$, the demodulation operation produces a processing gain approaching $10\log 10(T/\Delta t)$ dB.

For the purpose of common definition the processor, which may include a PLL, operates on a sliding block of $N$ complex data samples. The samples processed in the $m^{\text{th}}$ block are

$$k_m = mN(1 - O/N) + k \quad \text{for} \quad 0 \le k \le N-1, \qquad (2)$$

where $O$ is the number of offset samples, which can vary from 0 to $N$. The processor can also use it's current estimate of the reference signal, $v_r(k)$, for feedback constructed such that

$$v_r^*(k)v(k)/\left|v(k)\right| \simeq 1. \qquad (3)$$

For a phase locked loop $O = 0$ and $N$ is the order of the loop transfer function, which is typically less than 3. Direct frequency estimation uses much larger $N$ values with overlap. From a purely information theoretic perspective, using larger processing blocks should provide better frequency/phase estimates. The purpose of this note is to demonstrate this by direct comparison.

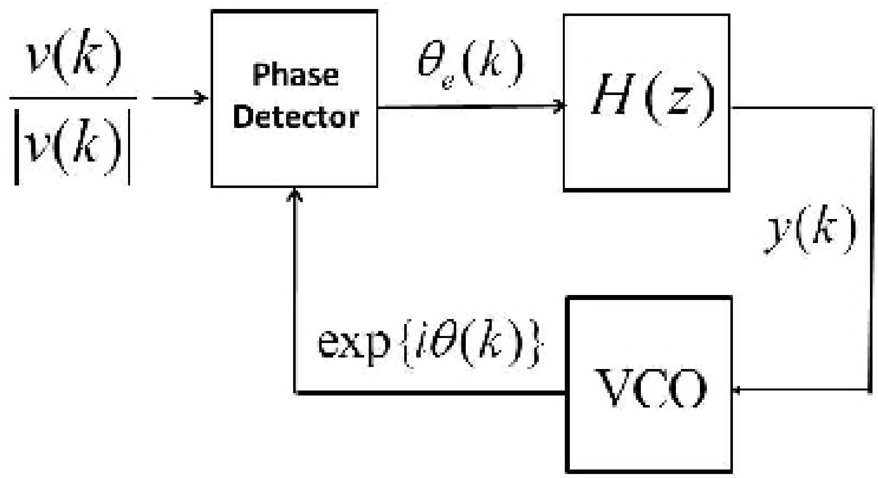


Figure 1: Functional diagram of PLL.

## 2 Phase Locked Loops

A phase locked loop (PLL) has three functional elements, namely a phase detector, a loop filter, and a voltage controlled oscillator (VCO) connected as shown in Figure (1). The current value of $v_r(k) = \exp\{i\theta_e(k)\}$ is the reference for the new input $v(k+1)/|v(k+1)|$.

To generate a realization of the complex data stream integration of the instantaneous frequency over the sample interval must be approximated. Two approximation follow:

$$\int_{t}^{t+\Delta t} \omega(t')\, dt' \simeq \left\{ \begin{array}{c} \omega(t)\, \Delta t \\ (\omega(t) + \omega(t+\Delta t))\, \frac{\Delta t}{2} \end{array} \right. \tag{4}$$

For signal generation the more accurate trapezoidal rule is used. For digital

VCO operation the integral is approximated by direct summation:

$$v_r(k) = \exp\left\{ -i \left( \omega_r k\Delta t + K \sum_{n=0}^{k-1} y(k)\Delta t \right) \right\}. \tag{5}$$

The feedback signal $y(k)$ is used to vary the oscillator frequency $\omega_r$.

The phase detector first multiplies the normalized input signal by the complex conjugate of the reference signal (5):

$$\begin{aligned} v_c(k) &= \frac{v(k)}{|v(k)|} v_r^*(k) \\ &= \exp\left\{ i \left( \sum_{n=0}^{k} \omega(k)\Delta t - K \sum_{n=0}^{k-1} y(n)\Delta t \right). \right\} \end{aligned}$$

The phase of $v_c(k)$ is defined by the arctangent operation

$$\theta_e(k) = \mathit{atan2}(\,\mathrm{Im}\,(v_c(k)), \mathrm{Re}\,(v_c(k))). \tag{6}$$

Digital IIR filters are defined by two sets of coefficients $A(n)$ and $B(n)$ such that

$$y(k) = \sum_{n=0}^{N-1} B(n)\theta(k-n) + \sum_{n=1}^{N-1} A(n)y(k-n) \tag{7}$$

The $A(n)$ coefficients act on previously computed outputs, which must be retained.

## 2.1 Software Implementation

The VCO phase includes a starting phase to offset the center frequency and a correction term

$$\theta_r(k) = \omega_r k\Delta t + K \sum_{n=0}^{k-1} y(k)\Delta t. \tag{8}$$

To isolate the starting frequency offset uncertainly let

$$\omega_r = \omega_0 - \delta. \tag{9}$$

The phase detector output is the difference between the input signal phase and the VCO phase;

$$\begin{aligned} \theta_e(k) &= \theta_v(k) - \theta(k) \\ &= \widetilde{\theta}_v(k) - K \sum_{n=0}^{k-1} y(k)\Delta t, \end{aligned} \tag{10}$$

where

$$\widetilde{\theta}_v(k) = \sum_{n=0}^{k} \omega(k)\Delta t + \delta k\Delta t. \tag{11}$$

Software implementation proceeds as follows:

1. $\theta(k) = \omega_r \Delta t k + \Delta\theta$
2. $v_r = \exp\{i\theta(k)\}$;
3. $v_c = v(k) * \texttt{conj}(v_r)$;
4. $\theta_e =$*atan2(*$\mathrm{Im}(v_c)$,$\mathrm{Re}(v_c)$);
5. Loop Filter $\theta_e \rightarrow y$
6. VCO Filter $y \rightarrow \Delta\theta$

The most commonly used configurations of the loop filter and the VCO are first-order filters with z transforms

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}. \tag{12}$$

Each filter implementation requires one storage register, $v_0$:

1. $v_1 = v_0$;
2. $v_0 = x_{\text{in}} - v_1 * a_1$;
3. $y_{\text{out}} = v_0 * b_0 + v_1 * b_1$;

The PLL operation is completely specified by the sample rate, $\Delta t$, the nominal VCO frequency, $\omega_r$, and the two sets of filter coefficients. The loop filter $H(z)$ and the VCO filter $N(z)$ can be implemented as a combined filtering operation $H(z)N(z)$, whereby the software implementation can be formulated as a single filtering operation.

Modern signal processing technology has made it possible to think of (1) as a defining relation. As a generic signal consider an infinite complex series $u(k)$. The two-sided $z$ transform is defined as

$$U(z) = \sum_{k=-\infty}^{\infty} u(k) z^{-k}. \tag{13}$$

Negative indices represent signal samples earlier that the reference time $k = 0$. The properties of the z-transform and its inversion follow from complex variable theory built on the Cauchy integral formula. Important properties include multiplication of z-transform to evaluate filtering operations such as (7). If the series is delayed by $n$ steps, the corresponding $z$ transform is multiplied by $z^{-n}$.

The frequency response of the filter is defined by the Fourier series obtained by the replacement $z = \exp\{i\omega k/\Omega\}$:

$$U_P(\omega) = \sum_{k=-\infty}^{\infty} u(k) \exp\{i\omega k/\Omega\}. \tag{14}$$

The function $U_P(\omega)$ is periodic over the radian frequency interval $-\Omega/2 < \omega < \Omega/2$. The periodic function can be formally identified with an aliased continuous transformation:

$$U_P(\omega) = \sum_{m=-\infty}^{\infty} U\left(\omega + m\Omega\right). \tag{15}$$

Ideally $U_P(\omega) \simeq U\left(\omega\right)$ for $|\omega| < \Omega/2$.

Optimum filter design could proceed directly from the $z$ and Fourier transforms. However, filter design generally proceeds from Laplace/Fourier transform space rather than the z/Fourier series representations that characterizes sampled data. The transformation

$$u/2 = \tan^{-1}(\pi\omega/\Omega), \tag{16}$$

maps the infinite frequency range $-\infty < \omega < \infty$ onto $-\pi/2 \leq u \leq \pi/2$. From the identity

$$s = \frac{\Omega}{\pi}\frac{\left(\exp(-iu/2) - \exp(-iu/2)\right)}{\exp(-iu/2) + \exp(-iu/2)}, \tag{17}$$

where $s = i\omega$, it follows that the transformation

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}, \tag{18}$$

where $T = 2\pi/\Omega$, maps the infinite frequency response onto the frequency range supported by the z transformation. Moreover, it can be shown that the left and right complex half planes map to the inside and outside the unit circle in the complex $z$ domain. Thus, non-linear (warped) mapping of a filter transfer function define over $|\omega| < \Omega/2$ to the corresponding range of $u$ via (16), preservers the desired frequency response. The phase response of the analogue filter is not preserved, but can be checked after the fact.

PLL analysis using continuous functions and filter operations would lead to the following Fourier-domain relations for the error signal and IIR filter:

$$\widehat{\theta}_e(\omega) = \widehat{\widehat{\theta}}_v(\omega) - K\frac{\widehat{y}(\omega)}{i\omega}. \tag{19}$$

and

$$\widehat{y}(\omega) = \widehat{F}(\omega)\widehat{\theta}_e(\omega). \tag{20}$$

Eliminating $\widehat{y}(\omega)$ it follows that

$$\widehat{\theta}_e(\omega)/\widehat{\widehat{\theta}}_v(\omega) = \frac{1}{1 + K\widehat{F}(\omega)/i\omega}. \tag{21}$$

The overall loop transfer function is defined as the VCO phase output divided by the input phase:

$$\begin{aligned} H(\omega) &= \widehat{\theta}(\omega)/\widehat{\widehat{\theta}}_v(\omega) \\ &= 1 - \left(K\widehat{y}(\omega)/i\omega\right)/\widehat{\widehat{\theta}}_v(\omega) \\ &= \frac{K\widehat{F}(\omega)/i\omega}{1 + K\widehat{F}(\omega)/i\omega}. \end{aligned} \tag{22}$$

Optimum filter design is based on the transfer function response and noise rejection. Formally, a constrained optimization maximizes the signal response

$$\int |H(\omega)|^2 \, d\omega / (2\pi), \tag{23}$$

while minimizing the noise response

$$\int |1 - H(\omega)|^2 \, d\omega / (2\pi). \tag{24}$$

Early PLL literature derived optimum analog transfer. However, most applications use a first-order proportional plus filter [1]. The loop filter is defined formally by two time constants, $\tau_1$ and $\tau_2$:

$$F_a(s) = (1/(s\tau_1) + \tau_2/\tau_1), \tag{25}$$

where $s = i\omega$. Applying the transformation (18)

$$F_z(z) = \frac{(C1 + C2) - C1z^{-1}}{1 - z^{-1}} \tag{26}$$

where

$$C1 = (\tau_2/\tau_1 - T/(2\tau_1)) \tag{27}$$

$$C2 = T/\tau_1 \tag{28}$$

The corresponding relations for the VCO integrator are

$$N_a(s) = K\frac{1}{s} \tag{29}$$

$$N_z(z) = K\frac{T}{2}\frac{1 + z^{-1}}{1 - z^{-1}}. \tag{30}$$

For software implementation the loop filter coefficients are

$$\begin{array}{cc} b_0^f & C1 + C2 \\ b_1^f & -C1 \\ a_1^f & -1 \end{array}$$

[1] http://www.liquidsdr.org/blog/pll-howto/

The VCO coefficients are

$$\begin{array}{cc} b_0^v & KT/2 \\ b_1^v & KT/2 \\ a_1^v & -1 \end{array}$$

The loop transfer function is

$$H_a(s) = \frac{2\varsigma\omega_n s + \omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}, \tag{31}$$

where

$$\omega_n^2 = K/\tau_1 \tag{32}$$

$$\varsigma = \tau_2\omega_n/2\ . \tag{33}$$

In terms of $\omega_n$ and $\varsigma$.

$$C1 = \left(2\varsigma\omega_n T - (\omega_n T)^2/2\right)/(TK) \tag{34}$$

$$C2 = (\omega_n T)^2/(TK) \tag{35}$$

Note that the loop gain in the VCO will cancel $K$. The defining parameters are $\varsigma$ and $\omega_n$. Figure 2 show the loop transfer function for $\varsigma = 0.7$ and $\omega_n = BW/10$. The two peaks occur at $\pm\omega_n$.

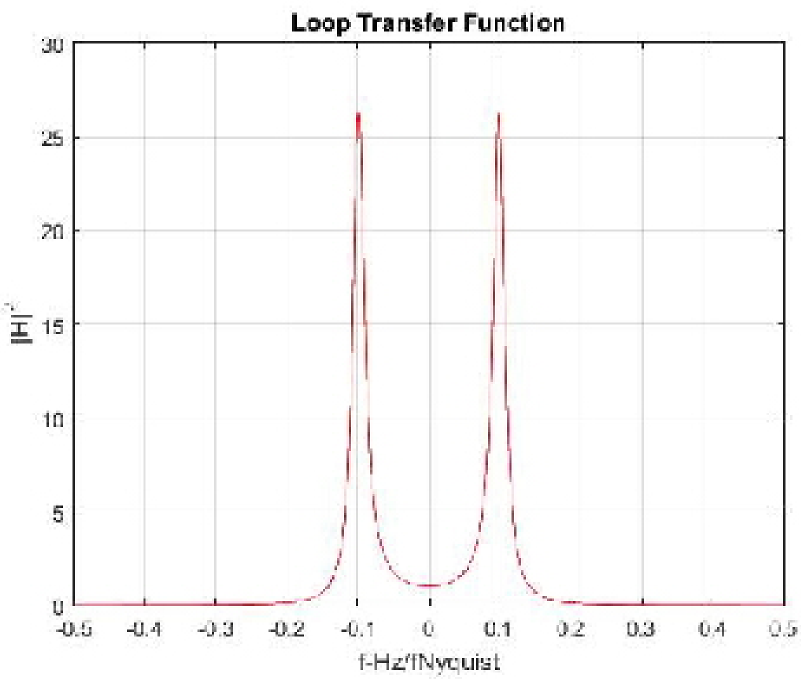


Figure 2: Loop transfer function for integration plus loop filter and integrator.

## 2.2 Software PLL Examples

To illustrate the SNR dependent PLL performance, a 50 kH signal as defined by (1) with $m(k) = 1$ was generated. A 10 kH frequency decreasing at 100 Hz per second was applied. Figure 3 shows a zoomed intensity display of a spectogram form with contiguous 2048 point unweighted periodograms. The full frequency range is -25 kHz to 25 kHz. The input SNR is 0 dB. The ~50 dB intensity peaks are due to the coherent processing gain achieved by the 2048 point Fourier transformations. The intensity variation is caused by the signal passing through the sinc functions that define the discrete frequency resolution. Figure 4 shows the second-order PLL results initiated with the correct frequency. The upper frame is derived from the loop phase by differentiation. The error curve is the difference between the derived frequency and the true frequency. Although the loop is initiated with the correct starting frequency there is a transient while the filter pipes up.

Figures 5 and 6 show the frequency hypothesis tracker (FHT) for SNRs of 10 and 0 dB. The FHT implementation used 2048 samples with 1024 point overlap. The FHT output is Doppler, but at a much coarser output than the sampling interval. The FHT tracker recovers the Doppler with better than 1 Hz accuracy for both the 10 and 0 dB SNR signals. The 0 dB PLL output does report a changing phase, but the error signals are outside the lock range.

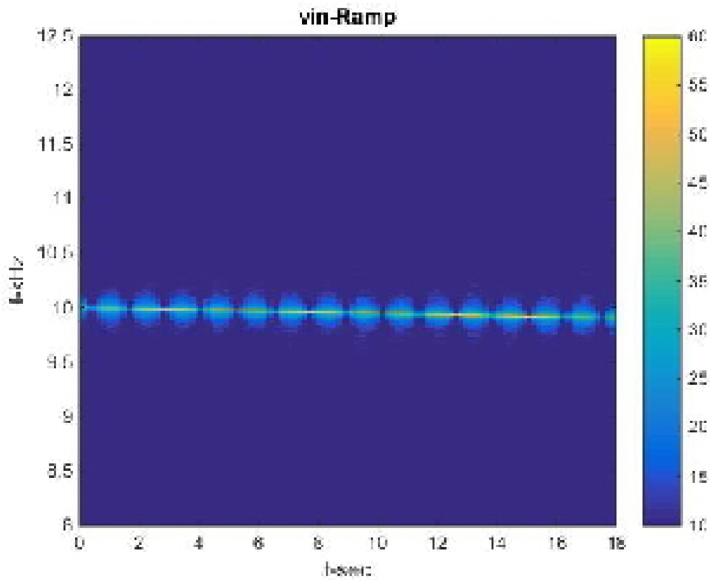


Figure 3: Spectogram of 10 kHz signal with a decrease of 100 Hz per second.

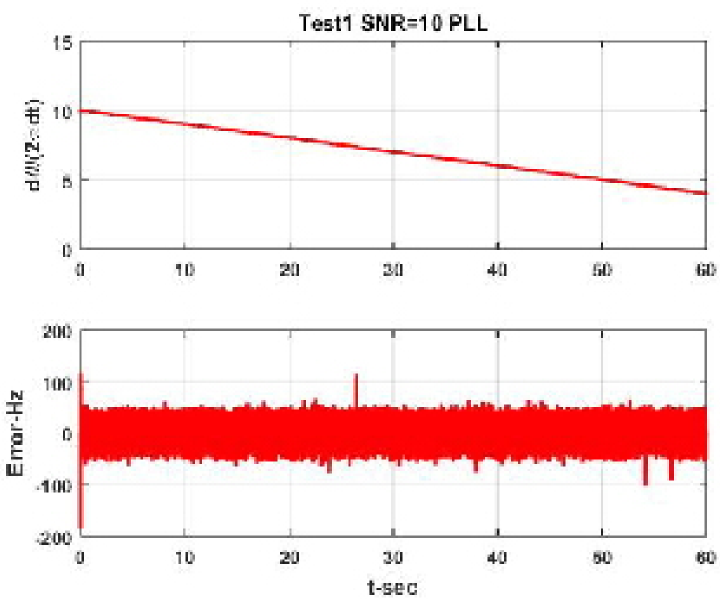


Figure 4: The upper frame is the derivitave of the PLL phase. The lower frame is the difference between the measured and true frequency.

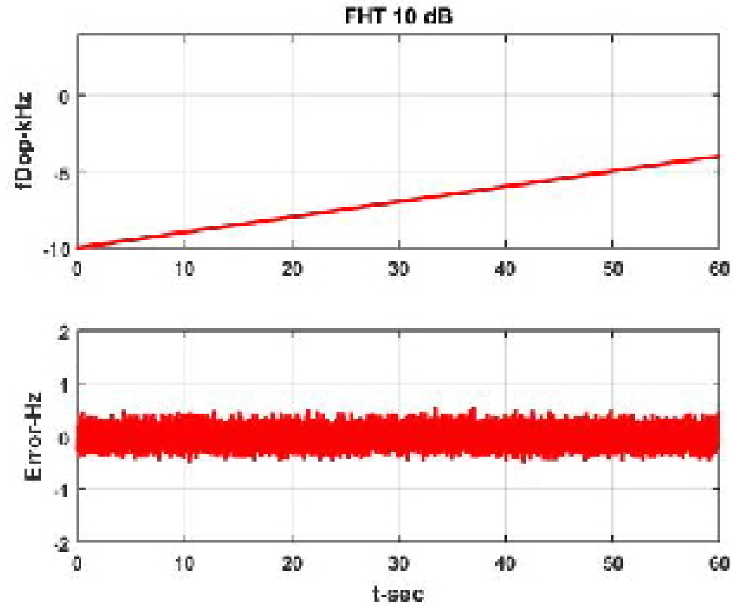


Figure 5: FHT Doppler and error for 10 dB SNR signal.

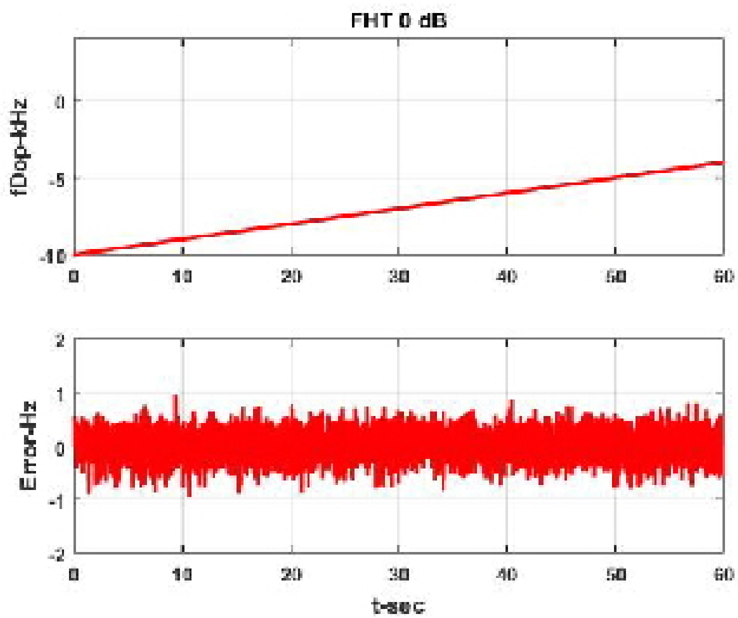


Figure 6: FHT Doppler and error for 0 dB SNR signal.

## References

[1] 49 papers. In William C. Lindsey and Marvin K. Simon, editors, *Phase-Locked Loops and Their Applications*. IEEE Press, 1978.

[2] Floyd M. Gardner. *Phaselock Technique*. John Wiley and Sons, Inc., New York, 1966.