Maximum Likelihood Iterative Parameter Estimation

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Abstract

Following a year long saga, I think we a better under standing of power-law parameter estimation in general and the Nelder-Mead simplex algorithm. Maximum likehood iterative parameter estimation (MLE-IPE) use the asymptitic χ_D distribution of the periodogram estimate of the spectral density function (SDF) or power spectral density (PSD) for time series. The parameters that define a theoretical SDFare systematically adjusted to maximize the likelihood that the SDF is representative of the process that generated the periodogram.

We have applied IPE to samples of in-situ diagnostics, which have two-component power-law SDFs and scintillation intensity SDFs, using a phase-screen model initiated by two-component power-law SDFs. Each application presents its own challenges. This note describes a common MatLab inplementation of MLE-IPE that can be used for both applications.

1 Background

Irregularity Parameter Estimation (IPE) adjusts parameters that define a two-component power-law SDF model to reconcile a the model with a an SDF estimate. The SDF estimate is an average of M periodogram estimates

$$\widehat{\Phi}_n^{(M)} = \frac{1}{M}\sum_m \widehat{\Phi}_n^{(m)}, \tag{1}$$

where the m superscript identifies an independent realization. If $F_k = F(k\Delta y)$, for $k = 0, 1, \cdots .N-1$, the periodogram estimate is defined as

$$P_n = \left|\widehat{F}_n\right|^2 /N \tag{2}$$

where

$$\widehat{F}_n = \sum_{k=0}^{N-1} F_k \exp\{iknk/N\}, \tag{3}$$

is the discrete Fourier transform. The natural-order spatial frequencies are defined as

$$q_n = [-N/2 \leq n \leq N/2 - 1]dq, \tag{4}$$

where $\Delta q = 2\pi/(N\Delta y)$. With the scaling

$$\widehat{\Phi}_n = \frac{N\Delta q}{2\pi} P_n = P_n/\Delta y, \tag{5}$$

$\left\langle \widehat{\Phi}_n \right\rangle = \Phi_n$, which shows that the scaled periodogram is an unbiased estimate of Φ_n.

For a broad class of homogeneous processes the PDF of $\widehat{\Phi}_n$ is asymptotically χ_{2M} distributed. If the frequency samples are independent, the log likelihood ratio can be computed as follows:

$$\begin{aligned} \Lambda(\widehat{\Phi}_n^{(M)}|\Phi_n) &= -\log \prod_{n=1}^{N} P_M(\widehat{\Phi}_n^{(M)}|\Phi_n) \\ &= \sum_{n=1}^{N} \Big[M(\widehat{\Phi}_n^{(M)}/\Phi_n) - M\log(M\widehat{\Phi}_n^{(M)}/\Phi_n) \\ &\quad + \log(\widehat{\Phi}_n^{(M)}) + \log\Gamma(M) \Big], \end{aligned}$$

where $P_M(\widehat{\Phi}_n^{(M)}|\Phi_n)$ is the conditional probability of $\widehat{\Phi}_n^{(M)}$ given that the underlying SDF is Φ_n.

We assume that Φ_n depends on a set of parameters

$$X_0 = [P_1, P_2, \cdots, P_J]. \tag{6}$$

Starting with an initial X_0 *guess,* irregularity parameter estimation (IPE) adjusts a subset of the parameters X_0,

$$X_1 = [P_{j\subset J,}], \tag{7}$$

to minimize $\Lambda(\widehat{\Phi}_n^{(M)}|\Phi_n)$. The dimension of X_0 is less than or equal to the J dimension of X_1. To identify the parameters that are being varied, the entries in X_1 that are being varied are replaced with NaNs. The fixed variables appear in X_1 explicitly. The combined vectors X_0 and X_1 define Φ_n, whereby

$$\Phi_n = \Phi(q_n|X_0, X_1). \tag{8}$$

That is, q_n and $\widehat{\Phi}_n^{(M)}$ together with X_0 and X_1 define $\Lambda(\widehat{\Phi}_n^{(M)}|\Phi_n)$. An initial specification of X_0 and X_1 defines the starting parameters and the parameters to be varied.

An IPE implementation adjusts the X_0 values is to minimize a prescribed objective function

$$f(X_0, \widehat{\Phi}_n^{(M)}, q_n, X_1) = \Lambda(\widehat{\Phi}_n^{(M)}|\Phi(q_n|X_0, X_1)). \tag{9}$$

The objective function defines the theoretical SDF as a function of the parameters defined by X_0 and X_1. For in-situ measurements, the two-component power law is defined by 4 parameters, namely C_p, η_1, η_2, and q_0:

$$\Phi(q|X_0, X_1) = C_p \left\{ \begin{array}{ll} q^{-\eta_1} & \text{for } q \le q_0 \\ q_0^{(\eta_1-\eta_2)} q^{-p_2} & \text{for } q \le q_0 \end{array} \right. . \tag{10}$$

For intensity SDFs, there are 5 parameters, U, p_1, p_2, μ_0, and ρ_F. The intensity SDF is computed with an algorithm developed by Carrano [1].

2 Nelder-Mead Simplex

The MatLab implementation of the Nelder-Meade Simplex is conveniently implemented with an implicit function call defining the objective function. Formally,

$$\begin{array}{c} f1 = @(X0)\texttt{objMLE\_IPE}(X0, qP, SDF\_P, X1, M, \cdots) \\ [X0, funF, exitFlag, \text{output}] = \texttt{fminsearch}(f1, X0, \text{options}) \end{array} ,$$

where $qP = [q_n]$ and $SDF\_P = \widehat{\Phi}_n^{(M)}$. The ellipsis includes parameters that provide options for specific implementations.

The behavior of the Nelder-Mead algorithm is acutely sensitive to how the objective behaves as parameters are varied about the true minimum and the starting parameters. The algorithms for direct power-law and scintillation intensity SDFs have tailored initiation procedures and control parameters.

2.1 MLE IPE for Two-Component Power-Law Processes

Parameter estimation for two-component power-law processes, include MLE, has been reviewed in the paper [2]. An intrinsic coupling between the turbulent strength and the low-frequency power-law index has been thoroughly investigated. It was found that an MLE procedure with $CsdB = 10\log_{10}(C_s)$ is substituted for C_s gives better convergence and an smaller parameter errors. Figure 1 summarizes the MLE-IPE estimates with $M = 1$ for 1000 realizations. Each MLE-IPE search was initiated with a log-linear least-squares estimate applied separately to large and small scale frequency ranges. Although the C_s parameter has a larger spread, the average is correct, $C_s = 10$. Figure 2 shows the exponential distribution that produces the average. Figure 3 shows scatter diagrams of the spectral index and turbulent strength parameters. While the η_1-C_s correlation is prominent in the scatter diagram it is not discernible in the error summaries.

2.2 MLE IPE for Intensity Spectra

To explore the IPE ramifications for intensity scintillation diagnostics, the N_e realizations were used to generate phase screens. The defining parameters for

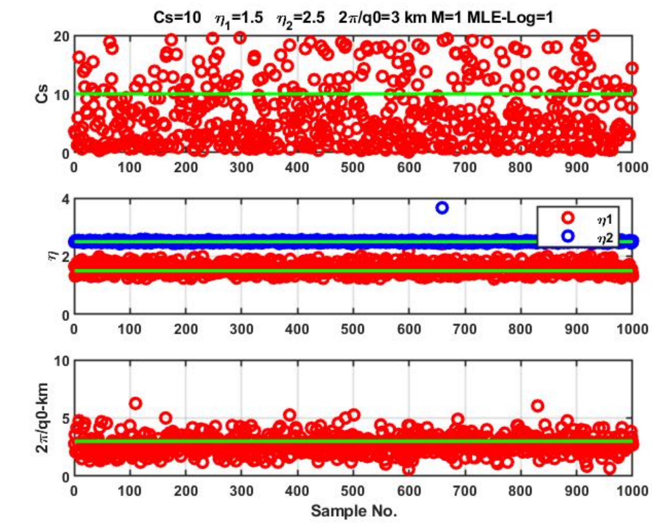

Figure 1: Figure 11 from [2].

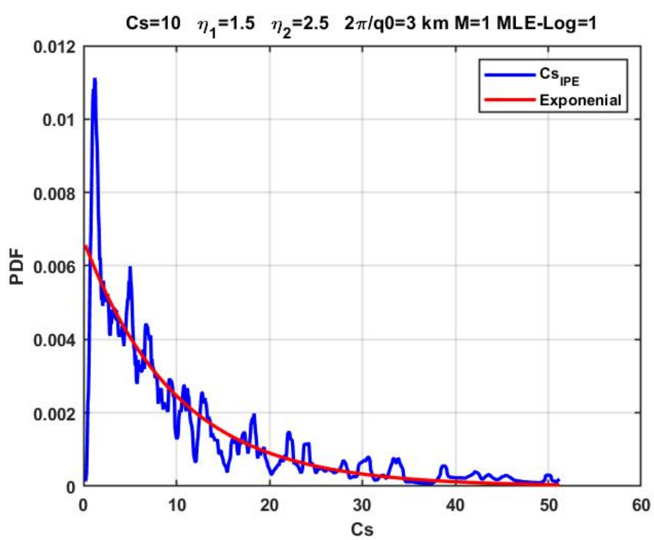

Figure 2: Figure 12 from [2].

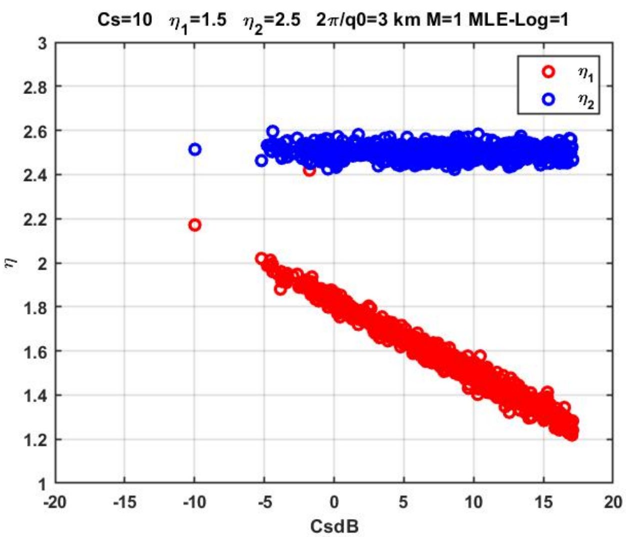

Figure 3: Figure 13 from [2].

the phase screen are C_p, p_1, p_2, and q_0. From [3], the relation between C_p and C_s is

$$C_p = (2\pi K/f)^2 \left[l C_s \right], \tag{11}$$

the path length, l, is absorbed in the definition of C_s. The remaining parameters convert total electron content to phase:

$$K = r_e c/(2\pi) \times 10^{16}, \tag{12}$$

where r_e is the classical electron radius, and c is the velocity of light.

The intensity SDF from the phase screen theory depends on parameters normalized to the Fresnel scale:

$$U = C_{pp} \begin{cases} 1 & \text{for} \quad \mu_0 > 1 \\ \mu_0^{p_2 - p_1} & \text{for} \quad \mu_0 < 1 \end{cases} \tag{13}$$

where $k = 2\pi f/c$, and

$$\rho_F = \sqrt{x/k} \tag{14}$$

$$\mu_0 = q_0 \rho_F \tag{15}$$

$$C_{pp} = C_p \rho_F^{p_1 - 1} \tag{16}$$

Figure 4 shows a GPS L1 frequency realization at a significant disturbance level. Figure 5 show the theoretical initiating phase screen and intensity SDFs (green). The blue curve is the periodogram of the phase-screen realization. The red curve is the periodogram of the upper frame in 4.

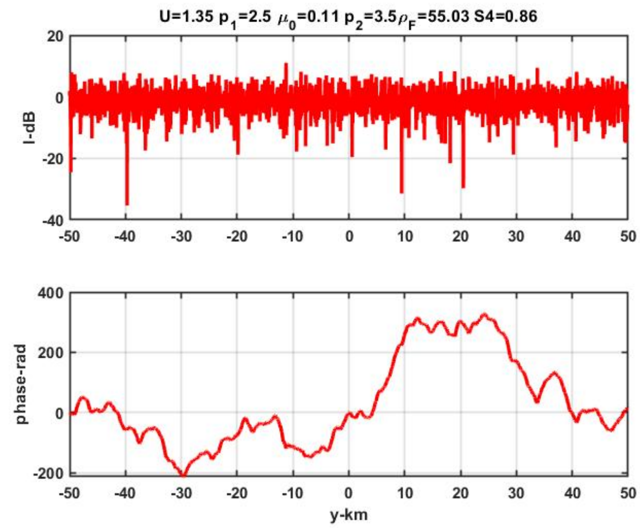

Figure 4: Phase screen realization at L1 GPS frequency with 100 km propagation from screen.

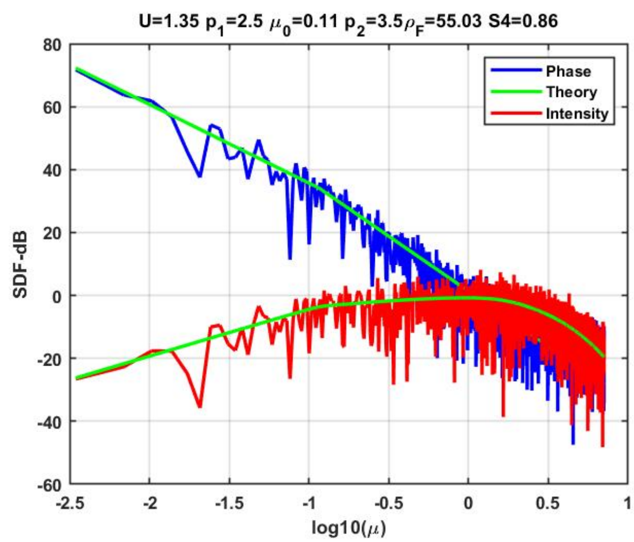

Figure 5: Green curves show theoretical phase-screen SDF and intensity SDF. Blue curve is periodogram of phase realization. Red curve is ;eriodogram of intensity realization.

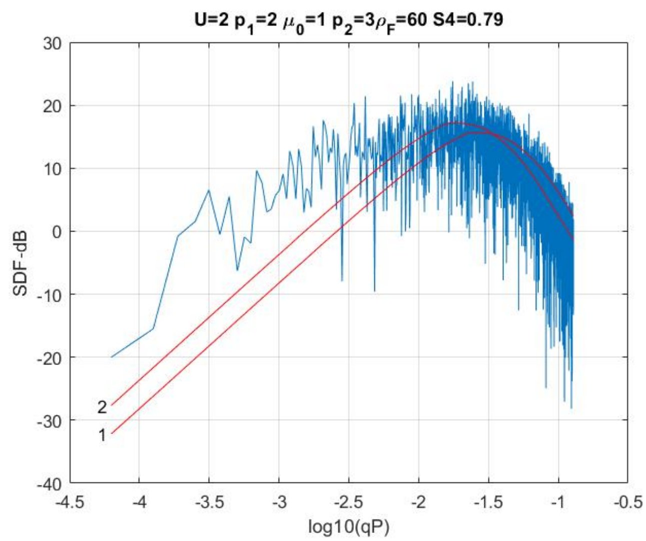

Figure 6: Example of interactive tool for manually adjusting intensity SDF parameters.

To make IPE work effectively, the initial guess must be reasonably close to the correct value. Knowing the scintillation index $S4$, constrains the U range. As an initial guess, we let $U = 1$ if $S4 \leq 0.5$ and $U = 2$ if $S4 > 0.5$, with $p_1 = 2$, $p_2 = 3$, and $\mu_0 = 1$. An initial value of ρ_F is obtained by aligning the `Ispectrum(U,p1,p2,mu0)` maximum value with the smoothed periodogram maximum value. Figure 6 shows the intensity periodogram (blue) with the aligned SDF overlaid as line 1 (red). An interactive utility `InitializeIParms4IPE` allows manual adjustment of the parameters. Line 2 shows the initial ρ_F value increased by an order of magnitude. The figure title lists the current U, p_1, p_2, μ_0, ρ_F and $S4$ values. The script might be a candidate for machine learning, for now manual trial and error must be used. With the starting values selected, and MLE-IPE search is initiated.

We found that a full 5-parameter search did not work well. A hybrid 4-parameter search with ρ_F fixed was more effective. We also found that truncating the search and restarting with the current set of values was more effective than waiting very slow convergence. To estimate the IPE parameter errors the estimated parameters from 100 trials is shown in Figure 7. The search was truncated at 50 iterations. The improvement with longer searches was ineligible. The U parameter, which is related to C_p, but not a direct measure of turbulent strength, has errors comparable to the other parameters. Figure 8, which is a scatter plot akin to Figure 3 shows no indication of correlation, which is attributed to the very different functional dependence on the intensity

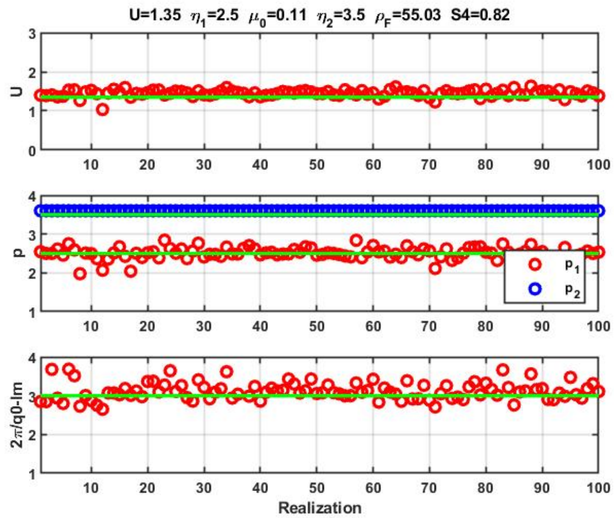

Figure 7: IPE errors for 4-parameter search initiated with correct parameters.

SDF features.

To illustrate an MLE-IPE result, 30 realizations were generated. With $M = 5$ a 4-parameter search was initiated with the starting value shown in Figure 6. From that point on, a 4-parametter search was initiated with the curren values. The result at the end of the 6^{th} realization is shown in Figure 9.

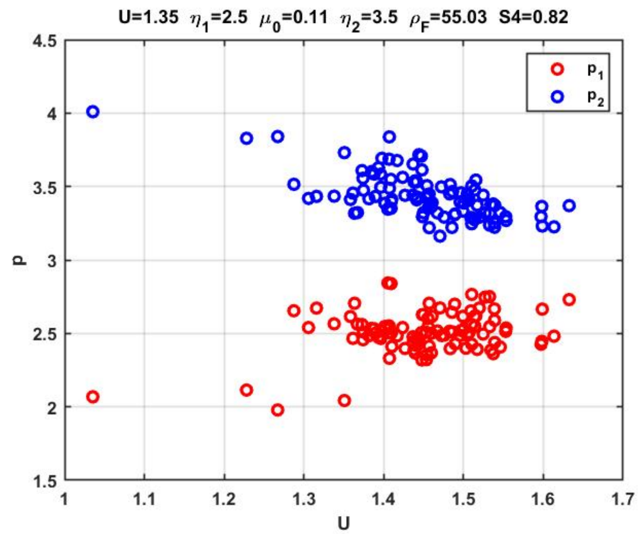

Figure 8: Scatter diagram of $p_{1,2}$ versus U for 100 4-parameter MLE-IPE runs.

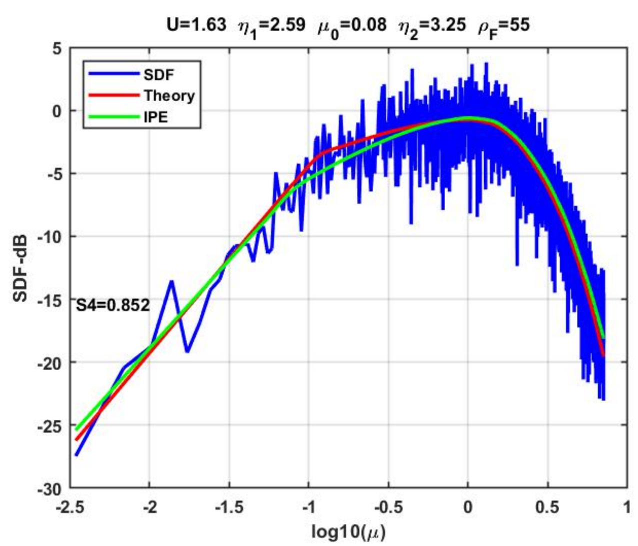

Figure 9: MLE-IPE example after 5 successive searches each initiated with previous values.

3 Discussion

We have developed a GPS/GNSS scintillation model based on a two-dimensional phase screen. The two-dimensional phase screen can generate multi-frequency relations that capture the frequency dependence and fading structure over the observed range of propagation disturbances. Model validation requires a demonstration that the model parameters are consistent with observations. IPE as developed by Charlie Carrano is the basis for the validation. The current results being pursued by Prof. Morton's students at the University of Colorado are very encouraging. However, IPE is a work in progress that has yet to be established as a reproducible procedure. Over the past year Charlie Carrano and I have been engaged in an intensive effort to cross check our respective implementations of Maximum-Likelihood IPE. I believe that process in now complete.

The MatLab software used to generate the examples in this report and the cited paper just submitted for publication is available on a shared Google Drive. The script `DemoIPE4PowerLawPhaseScreenIntensitySDFs.m` is intended to be a prototype for processing real data. Replace the simulated inputs with data inputs. The script `IPE4PowerLawSDFStats.m` will reproduce the power-law summary. The script `IPE4PhaseScreenStats.m` will reproduce Figure 9, but it needs interactive help and runs a long time. It writes a summary file. The script `SummarizeMLE_IPEparameters.m` will plot the results.

Two MatLab libraries have been generated. `Ispectrum` is a cleaned up version of the earlier library by the same names. It's the MatLab resource of using Charlie's software complied from his C++ code. `IPE_Utilities` contains the blessed IPE utilities. There is a lot of room here for continued development, particularly automating the manual adjustments.

References

[1] C. S. Carrano and C. L. Rino. A theory of scintillation for two-component power law irregularity spectra: Overview and numerical results. *Radio Sci.*, 51(doi:10.1002/2015RS005903):789âĂŞ813, 2016.

[2] C Rino, C Carrano, K Groves, and T Yokoyama. On the characterization of intermediate scale ionospheric structure. *Radio Science*, 2018. Submitted for publication.

[3] C Rino, B Breitsch, U Morton, Jiao, X Dongyang, and C Carrano. A compact multi-frequency gnss scintillation model. *Journal of the Institute of Navigation*, 2018. Accepted for publication.