A Software Defined Phase Locked Loop

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Abstract

This report summarizes a software defined phase locked loop and its connection to standard second-order loop design parameter for a integrating VCO and DDFS frequency synthesizer.

1 Introduction

Phase locked loops (PLLs) have been used since the early development of radio. Critical paper collections [1] and books [2] cover the subject matter, which includes nonlinear feedback and stochastic differential equations. Contributions have been made by information theory pioneers such as H. L. Van Trees and A. J. Viterbi. The subject matter can be intimidating, but software-defined radio makes it possible to explore PLLs with compact easily constructed simulations.

This report summarizes my own experience with software defined PLLs. I benefited from blogs posted by Paul Lutus 1 and Joseph D. Gaeddert 2. My implementation is structured only for software implementation. Input digital data streams are prerecorded. Paul Lutus’s development is structured for real-time digital processing. Gaeddert’s presentation is a Software implementation, with combines the phase-detector and loop-filter functions into a single filter.

2 PLL Review

A phase locked loop (PLL) has three functional elements, namely a phase detector, a loop filter, and a voltage controlled oscillator (VCO). Figure (1) shows these functional elements and their interconnections. The index $k$ identifies a time sample. Software defined PLLs processes pre-recorded critically sampled complex data sequences.

1http://www.arachnoid.com/phase_locked_loop/index.html
2http://liquidsdr.org/blog/pll-simple-howto/
Figure 1: Functional diagram of PLL.
2.1 Signal Model

PLLs are an integral part of communication and navigation receivers. Digitally sampled signals in clear propagation environments devoid of interference, multipath, and propagation disturbances can be modeled as

\[ v(k) = \sqrt{SNR(k)}m(k) \exp \left\{ i \left( \omega_0 k \Delta t + \int_0^kt \omega(t') dt' \right) \right\} + \zeta(k). \quad (1) \]

The first term in the exponential argument is the phase progression corresponding to a constant center frequency \( f_0 = \omega_0/(2\pi) \). The second term is the integral of a slowly changing instantaneous frequency, \( \omega(t) \). The signal amplitude is defined by the power signal-to-noise ratio, \( SNR(k) \), which includes path and system losses as well as antenna and amplifier gains. Knowledge of transmitted peak power and noise complete the SNR definition. The component \( m(k) \) represents modulation imparted at transmission. The average intensity of the modulation is constrained to unity. For performance evaluation only SNR need be considered. Critical sampling captures the modulation frequency content. The sampling bandwidth must capture the extremes of \( \omega(t) \) as well. The term \( \zeta(k) \) is a unit variance zero mean uncorrelated complex random sequence, usually Gaussian, representing receiver noise.

In the real world propagation disturbances introduce an additional stochastic modulation with unit average intensity. Signals sharing the same frequency, \( f_0 \), can be distinguished by transmitting different modulation sequences (e.g. GPS and cell phones). The operating environment is a source of multipath reflections of the transmitted signal. Mitigation of interference and multipath is largely a matter of antenna design and siting. The combined operations of demodulation and frequency tracking is complicated. Only the clear channel signal as defined by (1) with \( m(k) = 1 \) will be considered here.

2.2 Component Models

To generate a realization of the complex data stream integration of the instantaneous frequency over the sample interval must be approximated. Two approximation follow:

\[ \int_t^{t+\Delta t} \omega(t') dt' = \begin{cases} \omega(t) \Delta t, & \text{if } \omega(t) + \omega(t + \Delta t) \geq 0 \\ (\omega(t) + \omega(t + \Delta t)) \frac{\Delta t}{2}, & \text{if } \omega(t) + \omega(t + \Delta t) < 0 \end{cases} \quad (2) \]

For signal generation the more accurate trapezoidal rule can be used. For digital VCO implementations integration is approximated by direct summation:

\[ v_r(k) = \exp \left\{ -i \left( \omega_r k \Delta t + K \sum_{n=0}^{k-1} y(k) \Delta t \right) \right\}, \quad (3) \]
The term $y(k)$ is a feedback signal that offsets the oscillator frequency $\omega_r$. Note that $K$ has the units $t^{-1}$.

The phase detector first multiplies the normalized input signal by the complex conjugate of the reference signal $(3)$:

$$v_c(k) = \frac{v(k)}{|v(k)|} v_r^*(k)$$

$$= \exp \left\{ i \left( \sum_{n=0}^{k} \omega(k) \Delta t - K \sum_{n=0}^{k-1} y(n) \Delta t \right) \right\}. \tag{3}$$

The real-world normalization operation is an automatic gain control that keeps the input to the PLL at a constant average signal level. The phase of $v_c(k)$ is formally derived by an arc tangent:

$$\theta_c(k) = \text{atan}2(\text{Im}(v_c(k)), \text{Re}(v_c(k))). \tag{4}$$

The arc tangent keeps track of the complex plane quadrant of the phase.

Digital IIR filter are defined by two sets of coefficients $A(n)$ and $B(n)$ such that

$$y(k) = \sum_{n=0}^{N-1} B(n) \theta(k-n) + \sum_{n=1}^{N-1} A(n) y(k-n) \tag{5}$$

The filter implementation uses previously computed outputs and earlier signal samples, which must be retained and updated.

### 2.3 Software Implementation

The VCO phase includes a starting phase to offset the center frequency and a correction term

$$\theta_r(k) = \omega_r k \Delta t + K \sum_{n=0}^{k-1} y(k) \Delta t. \tag{6}$$

To isolate the starting frequency offset uncertainty let

$$\omega_r = \omega_0 - \delta. \tag{7}$$

The phase detector output is the difference between the input signal phase and the VCO phase:

$$\theta_c(k) = \theta_v(k) - \theta(k)$$

$$= \tilde{\theta}_v(k) - K \sum_{n=0}^{k-1} y(k) \Delta t, \tag{8}$$

where

$$\tilde{\theta}_v(k) = \sum_{n=0}^{k} \omega(k) \Delta t + \delta k \Delta t. \tag{9}$$
Hardware implementations use multipliers in place of the phase detector with amplification preceding the filtering operation. Additionally, the real and imaginary signal components must be treated explicitly.

Software implementation proceeds as follows:

1. \( \theta(k) = \omega_r \Delta t k + \Delta \theta \)
2. \( v_r = \exp \{i \theta(k)\} \)
3. \( v_c = v(k) \ast \text{conj}(v_r) \)
4. \( \theta_c = \text{atan2}(\text{Im}(v_c), \text{Re}(v_c)) \)
5. Loop Filter \( \theta_c \rightarrow y \)
6. VCO Filter \( y \rightarrow \Delta \theta \)

The loop filter and the VCO are first-order filters with z transforms of the form

\[
H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}.
\]  

(10)

Each filter implementation requires one storage register, \( v_0 \):

1. \( v_1 = v_0 \)
2. \( v_0 = x_{in} - v_1 \ast a_1 \)
3. \( y_{out} = v_0 \ast b_0 + v_1 \ast b_1 \)

The loop filter coefficients are defined as follows.

\[
\begin{align*}
b_0^f &= C_1 + C_2 \\
b_1^f &= -C_1 \\
\end{align*}
\]

Integrator VCO filter coefficients are

\[
\begin{align*}
b_0^v &= K \Delta t / 2 \\
b_1^v &= -K \Delta t / 2 \\
a_1^v &= -1.
\end{align*}
\]

Direct digital frequency synthesizer (DDFS) coefficients are

\[
\begin{align*}
b_0^v &= 0 \\
b_1^v &= -K \Delta t / 2 \\
a_1^v &= -1.
\end{align*}
\]

The PLL simulation is completely specified by the sample rate, \( \Delta t \), the nominal VCO frequency, \( \omega_r \), and the two sets of filter coefficients. Steps 1 through 6 are recursive.
3 Optimum PLL Design

3.1 Background

PLLs were developed when only analog realizations were feasible. Filters were defined by their frequency domain intensity response $|H(\omega)|^2$, usually expressed in dB. The filter phase had to be constructed in such a way that the impulse response $h(t) = 0$ for $t < 0$. The inverse Fourier transform that defines the impulse response of the filter can be calculated as a Fourier transform or a Laplace transform:

$$h(t) = \int_{-\infty}^{\infty} H(\omega) \exp\{i\omega t\} \frac{d\omega}{2\pi}$$

$$= \frac{1}{i} \int_{\sigma-i\infty}^{\sigma+i\infty} H(-is) \exp\{-st\} \frac{ds}{2\pi}.$$  (12)

The Laplace transform, with an offset $\sigma$ emphasizes the extension of the Fourier integral from the real axis to the complex plane. Laplace transforms that confine the poles of $H(-is)$ to the right or left-hand planes have non-zero impulse responses in the opposite half plane. Transfer functions of the form

$$H(s) = \frac{(s-z_0)(s-z_1)}{(s-p_0)(s-p_1)} \ldots$$

are particularly convenient.

A sampled function supports a Fourier series representation $a(k)$. The transfer function is the aliased periodic form of $H(\omega)$,

$$H_P(\omega) = \sum_{l=-\infty}^{\infty} H(\omega + l\Omega).$$

The Fourier series representation is

$$H_P(\omega) = \sum_{k=-\infty}^{\infty} a(k) \exp\{-2\pi i k \omega / \Omega\}$$

$$= \sum_{k=-\infty}^{\infty} a(k) z^{-k}$$

where

$$z = \exp\{-2\pi i \omega / \Omega\},$$

and

$$a(k) = \int_{-\Omega/2}^{\Omega/2} H_P(\omega) \exp\{-2\pi i k \omega / \Omega\} \frac{d\omega}{2\pi}.$$  (18)

The $z$-transform (16) extends the Fourier series representation to the complex plane in the same way that the Laplace transform (12) extends the Fourier
integral representation to the complex plane. By applying the $z$-transformation to (5) it follows that a realization of the filter can be implemented by identifying the Fourier coefficients with delays commensurate with the associated power of $z$.

At this point one could proceed by interpreting sampled digital functions as Fourier series coefficients. Whereas analog filters are characterized by their frequency response over the infinite interval $-\infty < \omega < \infty$, sampled functions have Fourier series representations over the finite frequency interval $-\Omega/2 \leq \omega_P \leq \Omega/2$. The transformation

$$ u/2 = \tan^{-1}(\pi \omega / \Omega), $$

maps $-\infty < \omega < \infty$ to $-\pi/2 \leq u \leq \pi/2$. From the identity

$$ \pi \omega / \Omega = \frac{-i (\exp(-iu/2) - \exp(-iu/2))}{\exp(-iu/2) + \exp(-iu/2)}, $$

it follows that

$$ i \omega = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}, $$

where $T = \pi / \Omega$ is the Nyquist sampling rate.

The transformation (21) is exact only to the extent that $H(\omega) = H_P(\omega)$ over the range $-\Omega/2 \leq \omega_P \leq \Omega/2$. One can show that (21) preserves causality by mapping poles in the appropriate infinite half plane to poles inside and outside the unit circle in the complex $z$ plane. It is perhaps noteworthy that a causal discrete filter can be constructed from a specification of $|H_P(\omega)|^2$ using only discrete Fourier Transforms (DFTs). The approximation to the equivalent analog filter can be constructed effectively by applying (21) [3].

### 3.2 PLL Transfer Function

Following the background discussion, assume that $\theta_e(k)$, $\tilde{\theta}_e(k)$, and $y(k)$ are Fourier series coefficients with equivalent periodic frequency domain representations $\tilde{\theta}_e(\omega)$, $\tilde{\theta}_v(\omega)$, and $\hat{y}(\omega)$. From (8) it follows that

$$ \tilde{\theta}_e(\omega) = \tilde{\theta}_v(\omega) - K \frac{\hat{y}(\omega)}{i \omega}. $$

The IIR filter admits the Fourier domain representation

$$ \hat{y}(\omega) = \hat{F}(\omega) \tilde{\theta}_e(\omega). $$

Eliminating $\hat{y}(\omega)$ it follows that

$$ \tilde{\theta}_e(\omega) \bigg/ \tilde{\theta}_v(\omega) = \frac{1}{1 + K \hat{F}(\omega) / i \omega}. $$

7
The overall loop transfer function is defined as the VCO phase output divided by the input phase:

\[ H(\omega) = \frac{\hat{\theta}(\omega)}{\hat{\theta}_v(\omega)} = 1 - \frac{(K\hat{p}(\omega)/i\omega)/\hat{\theta}_v(\omega)}{1 + K\hat{F}(\omega)/i\omega}. \]  

(25)

Optimum filter design is based on the transfer function response and noise rejection. Formally, a constrained optimization maximizes the signal response

\[ \int |H(\omega)|^2 d\omega / (2\pi), \]

while minimizing the noise response

\[ \int |1 - H(\omega)|^2 d\omega / (2\pi). \]

Optimum analog filters have been designed by using Laplace transforms, although a present day design could proceed directly without recourse to analog representations. To avoid repeating lengthy computations, it is prudent to use the analog design results as a guideline.

A first-order proportional plus integration (PI) filter has been used extensively. The analog transfer function and z transform representations follow:

\[ F(s) = \frac{1}{(s\tau_1) + \tau_2/\tau_1} \]  

(26)

\[ F(z) = \frac{(C1 + C2) - C1z^{-1}}{1 - z^{-1}} \]  

(27)

where

\[ C1 = \frac{\tau_2/\tau_1 - T/(2\tau_1)} \]  

(28)

\[ C2 = T/\tau_1 \]  

(29)

For a VCO integrator,

\[ N_v(z) = K\frac{T}{2} \frac{z^{-1}}{1 - z^{-1}}. \]  

(30)

Chung et. al [4] introduced the direct digital frequency synthesizer (DDFS) z-transform

\[ N_D(z) = K\frac{T}{2} \frac{z^{-1}}{1 - z^{-1}}. \]  

(31)

The closed-loop transfer functions can be written as

\[ H(s) = \frac{KF(s)N_a(s)}{1 + KF(s)N_a(s)} \]  

(32)

\[ H(z) = \frac{KF(z)N_a(z)}{1 + KF(z)N_a(z)} \]  

(33)
For the integrator VCO

\[
H^v(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},
\]

where

\[
\omega_n^2 = \frac{K}{\tau_1} \quad \zeta = \frac{T\omega_n}{2}.
\]

The loop filter coefficients are

\[
C_1^v = \left(2\zeta (\omega_n T) - (\omega_n T)^2 / 2\right) / (KT) = \frac{T}{\tau_1} - \frac{T}{\tau_1}
\]

\[
C_2^v = (\omega_n T)^2 / (KT) = \frac{T}{\tau_1}
\]

For the DDFS VCO Chung et al. [4] presented the loop filter coefficients

\[
C_1^D = \frac{1}{KT} \frac{8\zeta (\omega_n T)}{4 + 4\zeta (\omega_n T) + (\omega_n T)^2}
\]

\[
C_2^D = \frac{1}{KT} \frac{4 (\omega_n T)^2}{4 + 4\zeta (\omega_n T) + (\omega_n T)^2}
\]

which maintain the same closed-loop transfer function. In their version \(KT\) is replaced by \(K\), which evidently occurred because the unnormalized \((T = 1)\) form of (16) was used. As noted earlier, the gain factor \(K\) has \(t^{-1}\) units. The overall loop transfer function is independent of \(K\).

### 3.3 Software PLL Examples

The PLL described in 2.3 can be translated directly to MatLab. The intended application is processing VHF, UHF, and L-Band digitally recorded beacon satellite data, which recorded as 25 or 50 kHz complex data streams. The satellite range change causes a varying Doppler shift that can span a ±10 kHz range over the 15 to 20 minutes the satellite is visible to a ground based receiver. As a demonstration signal, 25 kHz complex signals were generated with a 10 kHz frequency varied by a negative linear frequency of 5 Hz per sec.

The loop parameters are completely defined by \(\zeta, \omega_n, \) and the sampling interval \(\Delta t\). The \(K\Delta t\) scaling in the definitions of \(C_1\) and \(C_1\) are canceled when the VCO filter is implemented. The values \(\zeta = 0.707\) and \(\omega_n = 0.1\) were used. Spectograms are commonly used to display a narrowband signal with a slowly varying signal. Figure 2 shows the spectogram of a 15 dB SNR complex signal. The spectogram is generated by concatenating 4096 point PSDs. The PSD coherent integration produces a 36 dB processing gain, which adds to the 15 dB SNR where the PSD maxima occur. The spectogram
Figure 2: Spectogram of 10 kHz signal with negative linear frequency variation.
intensity variation is caused by the convolution of the PSD frequency response with changing frequency.

Figure 3 shows the VCO phase with the linear reference phase subtracted to show the ability of the PLL to capture small phase variations. The lower frame shows the phase detector output, which is the feedback signal to be filtered and used to control the VCO frequency. If the SNR drops below 10 dB the PLL fails to lock the signal. The VCO phase has a smaller noise component because of the loop filter. However, the noise structure reflects the nonlinear feedback which is highly correlated from sample to sample. Figure 4 shows the same display for a 30 dB SNR signal. The DDFS filter has measurably better acquisition and noise characteristics, but no detailed comparisons have been made.
Figure 3: Loop VCO phase with linear phase removed (upper frame) and loop error (lower frame) for 15 dB realization.
Figure 4: Loop VCO phase with linear phase removed (upper frame) and loop error (lower frame) for 30 dB realization.
References


