A Configuration-Space Model for Intermediate-Scale Ionospheric Structure

Charles Rino¹, Charles Carrano¹, Keith Groves¹, and Tatsuhiro Yokoyama²

¹Institute for Scientific Research, Boston College
²National Institute of Information and Communications Technology, Tokyo, Japan

Key Points:

• Configuration-space models generate structure realizations by summing field-aligned striations as an alternative to imposing spectral characteristics on uncorrelated Fourier modes.
• The striation contributions are localized whereby parallelization can be used for efficient computation.
• Striation scales and peak electron densities can be constrained to support multi-component power-law spectral density functions.

Corresponding author: Charles Rino, crino@comcast.net
Abstract

Stochastic ionospheric structure is characterized by spectral density functions (SDFs), which are formally the average intensity of Fourier transformations of the electron density structure. Structure elongation along magnetic field lines is typically accommodated by constraining contours of constant spatial correlation to field-aligned ellipsoidal surfaces. Structure realizations are generated by imposing the square root of the SDF onto uncorrelated random Fourier modes. The approach has been used successfully for interpreting both in-situ and remote propagation diagnostics. However, the only connection to the field-aligned structures generated by the underlying instability mechanism is the correlation scale.

This paper introduces a configuration-space model that constructs realizations as summations of field-aligned elemental striations with a prescribed scale and peak electron density. We show that by choosing the contributing scales according to a bifurcation rule and imposing a power-law intensity scaling, the corresponding SDF closely approximates an inverse power law. Thus, configuration-space realizations can be structured to reproduce prescribed or measured SDFs from one-dimensional scans. Stochastic variation comes from an imposed random distribution of striation locations, which are defined by their intercept in a reference slice plane. To conform more directly to physics-based simulations, the striations can be interpreted as voids in the background electron density.

The model is designed to support propagation simulations with arbitrary propagation angles relative to the local magnetic field direction. Relations between between in-situ structure and diagnostic measurements as well as phase-screen equivalence can be explored.

1 Introduction

Global ionospheric models, such as the International Reference Ionosphere (IRI) Bilitza and Reinisch [2015], characterize the background ionosphere and its aggregate motion with deterministic functional representations. Accommodating the stochastic structure generated by unstable configurations requires a statistical characterization of the structure. The most commonly used measure of stochastic ionospheric structure is the spectral density function (SDF), which is formally the average intensity of spatial Fourier decompositions of structure realizations. Because three-dimensional ionospheric structure is highly elongated along the earth’s magnetic field-lines, definitive stochastic structure characterization is confined to two-dimensional slice planes penetrated by the field-aligned structure. Conventional ionospheric structure models approximate the anisotropy by constraining spatial coherence to highly elongated elliptical surfaces. Although these models have been used to interpret in-situ and propagation diagnostics for decades Yeh and Liu [1982], a more direct connection between structural details and the parameters derived from measured SDFs is desirable. This paper develops a configuration-space model that provides such a connection.

Recent results from high-resolution physics-based equatorial plasma bubble (EPB) simulations illustrate the structure characteristics. Figure 1, which is taken from Yokoyama [2017], shows a representative EPB structure that resulted from bottom-side zonal perturbations applied one hour earlier. The structure development is mapped along magnetic field lines that terminate at low altitudes in opposite hemispheres. The left frame shows the central meridional slice plane where quasi-deterministic field-aligned structure is manifest. The right frame shows the equatorial cross-field slice plane where the stochastic structure is manifest. The left frame in Figure 2 shows the equatorial-plane detail between 300 and 750 km. The right frame shows the cross-field integrated electron den-
High-resolution EPB simulations generate intermediate-scale stochastic structure spanning tens of kilometers to hundreds of meters. At scales approaching tens of kilometers ionospheric structure transitions from quasi-deterministic to stochastic. Structure smaller than the EPB resolution limit (≈ 300 m) is governed by spatially dispersive processes. Excluding these contributions from the intermediate-scale range isolates structure that is spatially invariant (frozen) over typical measurement intervals. Ideally, diagnostic measurements are one-dimensional scans of the three-dimensional structure illustrated schematically in Figure 1.

We start with the assumption that the two-dimensional cross-field SDF is isotropic. Analysis of the structure shown in Figure 2 supports the hypothesis. The two-dimensional
SDF is denoted $\Phi_{N_e}(q_L)$, where $q_L$ is the magnitude of the two-dimensional spatial frequency in the equatorial plane. In oblique planes contours of constant electron density are elliptical projections of the assumed cylindrically symmetric structures. The one-dimensional SDF of a diagnostic scan is determined by an integration of the form

$$\varphi_{N_e}(q_y) = \int \Phi_{N_e}(q_L) \frac{dq_x}{2\pi},$$

(1)

where $q_L = \sqrt{Aq_y^2 + Bq_xq_y + Cq_y^2}$. The coefficients $A$, $B$, and $C$ depend on the angle of the magnetic field from the reference axis. The standard modeling approach assigns an integrable analytic form to $\Phi_{N_e}(q_L)$. However, configuration-space realizations can be constructed by assigning a target one-dimensional form to $\varphi_{N_e}(q_y)$ directly. Following the development in Carrano and Rino [2016], the following one-dimensional analytic SDF will be used:

$$\Phi_{N_e}(q) = C_s \left\{ \begin{array}{ll}
q^{-\eta_1} & \text{for } q \leq q_0, \\
q^{-\eta_2} & \text{for } q > q_0.
\end{array} \right.$$  

(2)

In (2), $q$ is the spatial frequency in radians per meter, $q_0$ is the spatial frequency at which the power law index transitions from $\eta_1$ to $\eta_2$, and $C_s$ is the turbulent strength. If $\eta_1 = \eta_2$, a single inverse power law applies over the as yet unspecified range of spatial wavenumbers. If $\eta_1 = 0$, the break scale can be interpreted as a conventional outer scale.

A formal relation between (2) and the one-dimensional path-integrated phase SDF is readily established. The orientation of the local coordinate system within which diagnostic measurements are defined is arbitrary, whereby there is no loss of generality in aligning the $x$ axis with the direction of integration. The path-integrated electron density is defined by the integration

$$\phi(y) = \int_L N_e(x,y) dx,$$

(3)

where $L$ is the length of the integration path. One can show that the one-dimensional SDF of $\phi(y)$ is

$$\varphi_\phi(q_y) = L \int \frac{\sin^2(q_x L/2)}{(q_x L/2)^2} \Phi_{N_e}(q_x, q_y) \frac{dq_x}{2\pi}.$$  

(4)

There are two limiting cases:

$$\varphi_\phi(q_y) = \left\{ \begin{array}{ll}
L \Phi_{N_e}(0, q_y) & \text{for } L >> \sigma, \\
L \int \Phi_{N_e}(q_x, q_y) \frac{dq_x}{2\pi} & \text{for } L << \sigma,
\end{array} \right.$$  

(5)

where $\sigma$ is a measure of the correlation scale along the integration path. Only the first limiting form, which is generally assumed for interpreting propagation diagnostics, provides a direct analytic relation.

We note that the Carrano and Rino [2016] model development assigns the analytic form (2) to the path-integrated phase. The turbulent strength $C_s$ is replaced by $C_p$, and the spectral indices, $\eta_m$, are replaced by $p_n$. The redefinition of $C_s$ absorbs the radio-frequency-dependent translation of total electron content (TEC) to phase. To the extent that $L >> \sigma$, $p_n = \eta_n + 1$. Correlation along the integration would lead to a path-integrated index closer to the in-situ one-dimensional index. The configuration space-model can be used to verify the relation between the in-situ model parameters and diagnostic measurements.

### 2 Configuration-Space Models

Magnetic field models generate geographic coordinates that trace magnetic field lines. Let $\zeta_1 \cdots \zeta_k$ represent distance along the $k^{th}$ field line from a reference point $\zeta^k$ at $r_k$. Initiation points are placed in a reference slice plane, ideally the equatorial plane.
Vector components parallel and normal to the field line in the plane of curvature, namely \( \zeta^k_s \) and \( \zeta^k_\perp \), can be constructed at any point along a field line. Curvilinear magnetic field coordinates are defined by the orthogonal vectors \( \zeta^k_s \) and \( \zeta^k_\perp \times \zeta^k_s \). Striations shapes are defined by profile functions, which are monotonically decreasing functions of either distance along the field line, \( \zeta \), or radial distance, \( |\zeta| \). At the reference point \( p_s(0) = p_\perp(0) = 1 \).

The defining equation for a striation is

\[
\Delta N_k(\zeta_s, \zeta_\tau) = F_k p_s(\zeta_s - \zeta^k_s) / \sigma_s p_\perp(\zeta_\tau - \zeta^k_\tau) / \sigma_k,
\]

where \( F_k \) is the striation peak electron density, \( \sigma_k \) is the cross-field scale, \( \zeta^k_\perp \) identifies the point of initiation. A configuration-space realization is constructed by summing the contributions from each striation:

\[
\Delta N(\zeta_s, \zeta_\tau) = \sum_{k=1}^{N_s} F_k p_s(\zeta_s - \zeta^k_s) / \sigma_s p_\perp(\zeta_\tau - \zeta^k_\tau) / \sigma_k,
\]

where \( N_s \) is the number of striations. The initiation points are constrained to lie in a cross-field plane. Extension of each striation along field lines populates a prescribed data space.

### 2.1 Statistical Characterization

To generate a statistically homogeneous realization, the striation initiation points are drawn from a uniform distribution. With \( \zeta^k_s \) confined to a reference slice plane, (7) simplifies to

\[
\Delta N(\zeta_s, \zeta_\tau) \simeq \sum_k F_k p_\perp (|\zeta_\tau - \zeta^k_\tau| / \sigma_k).
\]

The approximation assumes that \( p_s(\zeta_s - \zeta^k_s) / \sigma_s \approx 1 \) over the slice plane. Two-dimensional Fourier transforms over oblique slice planes can be computed analytically. Let \( \rho_s \) represent the projection of \( [\zeta_s, \zeta_\tau] \) onto an oblique slice plane. The two-dimensional Fourier transform of \( \Delta N(\rho_s) \) is a summation of the two-dimensional transforms of the projected striation profiles:

\[
\delta N(\kappa_s) \simeq \sum_k F_k \tilde{p}^{(2)}_\perp(\kappa_s \sigma_k) \exp \{-i \kappa_s \rho^k_s\},
\]

where

\[
\tilde{p}^{(2)}_\perp(\kappa) = \int p_\perp(\eta) \exp \{-i \kappa \cdot \eta\} d\eta
\]

is the two-dimensional Fourier transform of the projected profile function. If the slice planes are normal to the fields lines \( \rho \) and \( \kappa \) measure position and frequency magnitudes directly. For oblique slice planes, \( \rho^k_s \) and \( \kappa^k_s \) are quadratic forms representing the projection of the striation cross section and its mapping to the spatial Fourier domain.

The position of each striation imposes a linear phase shift in the spatial Fourier domain. The expectation SDF can be computed as

\[
\left\langle |\Delta N(\kappa_s)|^2 \right\rangle = \sum_k \sum_{k'} F_k \tilde{p}^{(2)}_\perp(\kappa_s \sigma_k) F_{k'} \tilde{p}^{(2)*}_\perp(\kappa_s \sigma_k) \times \exp \{-i (\kappa_s \cdot (\rho^k_s - \rho'^{k'}_s))\}.
\]

A uniform distribution of \( \rho^k_s \) and \( \rho'^{k'}_s \) over a sufficiently large area imposes the following formal constraint:

\[
\left\langle \exp \{-i \kappa_s \cdot (\rho^k_s - \rho'^{k'}_s)\} \right\rangle \propto \delta \left( \rho^k_s - \rho'^{k'}_s \right),
\]

where \( \delta (\cdot) \) is a delta function. Summing the non-zero contributions leads to an analytic representation of the two-dimensional SDF in any slice plane that intercepts field lines:

\[
\left\langle |\Delta N(\kappa_s)|^2 \right\rangle \propto \sum_k F_k^2 \left| \tilde{p}^{(2)}_\perp(\kappa_s \sigma_k) \right|^2.
\]
A similar calculation for a one-dimensional SDF will show that
\[
\left\langle |\Delta N(\kappa_s)|^2 \right\rangle \propto \sum_k F_k^2 \left| \tilde{p}_\perp^{(1)}(\kappa_s \sigma_k^2) \right|^2,
\]  
(15)
where \( \tilde{p}_\perp^{(1)}(\kappa_s \sigma_k^2) \) is the one-dimensional transform of the radial profile function. These relations apply as long as the striations are distributed uniformly over a sufficiently large data space.

### 2.2 Successive Bifurcation

To complete the model specification, the number of striations, their peak density, and their scale must be specified. The SDF defines the intensity of each spatial Fourier component. The configuration-space parameters \( F_k \) and \( \sigma_k \) perform similar defining functions in the spatial domain. The assignment is guided by **successive bifurcation**, which is often invoked to describe the EPB structuring process. Local enhancements with steepening gradients bifurcate by generating local density depletions. The flanking enhancements bifurcate similarly, forming a structure cascade. This is illustrated in Figure 3, which shows 4 consecutive 10-s zoomed images of the initial development of the central EPB structure in the left frame of Figure 2. A more detailed discussion can be found in Yokoyama et al. [2014].

![Figure 3](image_url)

**Figure 3.** Numbered frames show zoomed views of the central EPB that generated the structure shown in 2at 10-sec intervals.

Successive bifurcation is captured by the following relations for contributing striation scales, \( \sigma_j \), and, \( N_j \), the number of striations with scale \( \sigma_j \):
\[
\sigma_j = \sigma_{\text{max}} 2^{-(d-J_{\text{max}}-j)}
\]
\[
N_j = 2^{d-j} \quad \text{for} \quad j = 1, 2, \cdots, J_{\text{max}}.
\]  
(16)

The parameter \( d \geq J_{\text{max}} \) determines the number of striations at the largest scale, namely \( 2^{(d-J_{\text{max}})} \). If \( d = J_{\text{max}} \), there is only one striation at the largest scale. Each smaller striation has twice as many striation contributions. The total number of striations is the sum of the number of striations at each scale:
\[
N_s = \sum_j N_j.
\]  
(17)
To complete the configuration-space definition we let

$$F_k = C \sigma_k^\gamma.$$  \hspace{1cm} (18)

where $C$ and $\gamma$ are parameters to be determined. We note in passing that the discrete scales represented by $\sigma_j$ are identical to scales associated with discrete wavelet decompositions Mallat [2009].

Substituting (16) and (18) into (15) defines the expectation one-dimensional SDF:

$$\left\langle |\Delta N(\kappa_s)|^2 \right\rangle = C \sum_{k=1}^{N_s} \sigma_k^{2\gamma} \left| \hat{p}_\perp^{(1)}(\kappa_s) \right|^2.$$  \hspace{1cm} (19)

It is understood that $\sigma_k$ contains the $N_j$ repetitions of $\sigma_j$ for each contributing scale as defined by (16). Formally, (19) is an analytic expression for the expectation of the one-dimensional SDF of the configuration. From numerical evaluation of (19) we find that the parameter $\gamma$ and the power law indices are related as

$$\eta = 2\gamma + 2.$$  \hspace{1cm} (20)

The parameter $C$ is an overall scale factor that can be chosen so that the variance of each realization is equal in expectation to the integral of the target SDF.

Two-component power-law SDFs are accommodated by assigning $\gamma$ to contiguous groups of scales:

$$\left\langle |\Delta N(\kappa_s)|^2 \right\rangle = C_1 \sum_{k=1}^{k_b} \sigma_k^{2\gamma_1} \left| \hat{p}_\perp^{(1)}(\kappa_s) \right|^2 + C_2 \sum_{k=k_b+1}^{N_s} \sigma_k^{2\gamma_2} \left| \hat{p}_\perp^{(1)}(\kappa_s) \right|^2,$$  \hspace{1cm} (21)

where $k_b$ is the frequency at which the power-law index changes from $\eta_1$ to $\eta_2$. The parameters $C_1$ and $C_2$ control the overall scale and maintain the target SDF continuity at the transition spatial frequency $\kappa_0$.

The profile-function defines the radial decay of the striation. For computational efficiency the profiles should have finite extent. However, for good SDF definition, the spectral-domain sidelobes must decay faster than $s^{-2}$. We find that the raised cosine

$$p_\perp(\rho) = \begin{cases} 
(1 + \cos (2\pi \rho))/2 & \text{for } \rho < 1 \\
0 & \text{for } \rho \geq 1
\end{cases},$$  \hspace{1cm} (22)

provides a good compromise.

### 2.3 Configuration-Space Realization

Generating a configuration-space realization starts with a specification of the largest scale, $\sigma_{\text{max}}$, and the number bifurcations, $J_{\text{max}}$, which establish the contributing scale range. An adequate scale range can be checked by comparing an evaluation of (21) with the target SDF (2). The choice of $d > J_{\text{max}}$ determines the total number of striations to be randomly located over the defining plane. Each point in the data space should have contributions from one or more striations. Specifying the remaining parameters, $C_s$, $\eta_n = (\gamma_n - 1/2)$, and $q_0$ define the target SDF. Trial and error adjustment of $d$ and $J_{\text{max}}$ be used to improve the fit to the target SDF.

A two-dimensional realization in the cross-field plane is constructed by using mesh grid coordinates $X$ and $Y$, which are $N_x$ by $N_y$ matrices that define each point in the data space. In mesh-grid coordinates (21) can be written as

$$\Delta N(X,Y) = \sum_{k=1}^{N_s} C(k)p_\perp \left( \sqrt{(X + \eta_Xk)^2 + (Y + \eta_Yk)^2}/\sigma_k \right) \sigma_k^{\gamma(k)}.$$  \hspace{1cm} (23)
The variables $\eta_X^k$ and $\eta_Y^k$ are random offsets. As before, it is understood that $C(k)$, $\gamma(k)$, and $\sigma(k)$ capture the bifurcation rules and scaling. Mesh grid sampling and extent determine the spatial wavenumber range, which should capture the spatial frequencies corresponding to the largest and smallest scales.

A representative example with $J_{\text{max}} = 9$, $d = 13$, and $\sigma_{\text{max}} = 50$ km has been constructed. For the defining parameter selection, $N_s = 8176$ and $\sigma_{\text{min}} = 195.31$ m. A 500 km by 200 km $XY$ mesh grid with $N_x = 2048$ and $N_y = 4096$ samples was generated. The cell dimensions are $\Delta x = 24.4$ and $\Delta y = 48.8$. Figure 4 is a color density map of a realization with two-component power-law parameters, $C_s = 1$, $\eta_1 = 1.5$, $\eta_2 = 2.5$, and $\phi_0 = 2\pi/3000$. Figure 5 verifies the SDF characteristics. The magenta curve is the target SDF as defined by (2). The red curve is the theoretical SDF as defined by (21). The blue curve is the average of 2048 periodogram estimates derived from realization $y$ scans.

A finer mesgrid sampling will extend the frequency range to reveal the sidelobes of the smallest contributing striations. Similarly, a larger meshgrid will resolve frequencies corresponding to scales larger that $\sigma_{\text{max}}$ where the measured and theoretical SDF achieves a constant value. Statistical equivalence of configuration-space realizations implies a common distribution of characteristic scales. The distinctive EPB structure captured in 2 imparts definitive phase relations to the spatial frequencies, which are discarded in the SDF computation.

![Configuration-space realization of two-component power-law SDF](image)

**Figure 4.** Configuration-space realization of two-component power-law SDF.

The extension of (23) to three dimensions is straightforward in regions where the magnetic field lines can be approximated by parallel lines. For an arbitrary rectangular region defined in a topocentric coordinate system, the magnetic field direction can be specified by the unit vector

$$\mathbf{u}_B = -[\cos \theta_B, \sin \theta_B \cos \phi_B, \sin \theta_B \sin \phi_B].$$

(24)

Appendix A.3 in Rino [2011] derives the $3 \times 3$ rotation matrix $C$ that transforms a vector in the the topocentric system to a magnetic-field-aligned system:

$$\zeta = Cr.$$  

(25)

The angle $\phi_B$ defines the angle of the magnetic meridian plane with respect to the $xz$ plane. The angle $\theta_B$ is the angle with respect to the $x$ axis, $\sim -90$ at the magnetic equator, $\sim 0$ approaching the magnetic pole. The $x$ axis is the propagation or scan reference.
Figure 5. Summary of one-dimensional SDFs for two-component power-law realization and average of one-dimensional $y$ SDF estimates.

direction. Variations along the field lines are ignored, whereby the striations are truncated at the data-space boundaries. As already noted, field-aligned curvature and variation can be accommodated.

3 Summary and Conclusions

This paper developed and demonstrated a configuration-space model. In its current form the model populates a representative data volume with parallel field lines. Two-component power-law SDFs are realized by using a bifurcation rule to populate a defining slice plane such as the equatorial plane shown in Figure 1. The orientation of the magnetic field is arbitrary for a data volume defined by a propagation path or diagnostic probe scan. The current implementation imposes no variation along the direction of the magnetic field. The size distribution in any slice plane perpendicular to the field lines is identical. Structure differences in slice planes normal to the reference axis are manifestations of oblique field-line extensions. A cross-field scan generates stochastic variation. Translating propagation paths parallel to the magnetic field also generate stochastic path-integrated structure. However, path-integration greatly enhances propagation effects. The ramifications for propagation are being pursued and will be reported in subsequent publications.

The SIGMA model developed by Deshpande et al. [2014] is designed to simulate fully three-dimensional propagation through intermediate-scale structure. Stochastic SIGMA structure realizations are generated by imposing the amplitude of the desired SDF onto uncorrelated Fourier components. With an appropriately constrained SDF, the realizations are statistically similar to configuration-space realizations. However, configurations comprised of striations are texturally different, which may have ramifications for interpreting propagation and in-situ diagnostics. Three-dimensional Fourier transformations connect every Fourier mode to each point in the data space. In the configuration space model, the contributions to any point in the data space from each striation are independent, whereby parallel processing can be exploited. The realization shown in Figure 4 was executed with 20 min of computation using 6 processors on a high-end PC.

However realizations of higher-dimensional intermediate-scale structure are obtained, they provide an opportunity to evaluate diagnostic measurements. In the introduction
the connection to propagation diagnostics, which respond to path-integrated structure, was discussed. We showed that the SDF of path-integrated structure depends on the correlation of the structure along the integration path. Figure 6 shows the SDF of the $x$-direction path-integrated structure, which is necessarily a single realization (red), overlaid on the average one-dimensional SDF. In the smaller spatial frequency range the path-integrated and average one-dimensional SDFs are similar. In the higher frequency range the path-integrated one-dimensional SDF falls below the average one-dimensional SDF.

The behavior can be understood from (5). The low-frequency range is comprised of large-scale structure approaching 50 km, which introduces high correlation over the 50-km integration path. The path-integrated SDF reflects the structure through direct integration. At the higher frequencies, the smaller structure scales are decorrelated along the integration path. In that case the path-integrated SDF is a projection of the two-dimensional structure, which falls off more steeply. Although the relation is more easily understood in terms of configuration-space structure, the same behavior would be expected with conventional realizations.

In addition to connecting diagnostic measurement with more robust structure models, configuration-space realizations provide a test of a critical assumption for irregularity parameter estimation. The defining parameters are estimated by using a goodness-of-fit measure applied to periodogram estimates of the one-dimensional SDF. More recent results show that improvements can be realized by using maximum-likelihood estimation, which requires prior knowledge of the probability density function of the SDF estimator Carrano et al. [2017]. For realizations generated from linear superpositions of independent Gaussian Fourier components, the statistics of a periodogram estimator is exponential. However, it known that the periodogram SDF estimate is asymptotically exponential for a broad class of statistically homogeneous processes Kokoszka and Mikoschb [2000]. Configuration-space realizations provide an opportunity to demonstrate the asymptotic exponential relation. Figure 7 shows the variation of the SDF estimates about the nominal mean. The second and third fractional moments,

$$F_m = \frac{\langle S\hat{D}F^m \rangle}{\langle S\hat{D}F \rangle^m}$$

for an exponential distribution $F_2 = 2$ and $F_3 = 6$. The measured moments are show in the figure. The measured moments are indeed close to the theoretical values for an exponential distribution even though the electron density structure characteristics are very different.
**Figure 6.** Path-integrated SDF (red) compared to average one-dimensional SDF (blue)

**Figure 7.** Variation of SDF estimates from configuration-space realization.

**Acknowledgments**

The reported research was supported in part by AFRL contract FA9453-12-C-0205, "ADVANCED DATA DRIVEN SPECIFICATION AND FORECAST MODELS FOR THE IONOSPHERE-THERMOSPHERE SYSTEM" All the results shown in the paper are simulations derived as described in the paper.

**References**


